New Curriculum
New Opportunities

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FOREWARD

The theme for the Conference '*New Curriculum – New Opportunities*' has encouraged Authors to contribute to new and different aspects of the curriculum taking the best of the past and adapting it for the needs of the future as well as taking advantage of the new technologies.

Topics have ranged from pedagogy to practical, from modelling to maths in context, from perceptions to school performance and whilst many articles were written by academic researchers and teacher educators, it is gratifying to see how many classroom teachers have contributed.

This year has unfortunately seen a reduction in the number of papers. But the year has also seen a different editorial panel mainly consisting of practicing teachers from Primary and Secondary schools, Government and Private , rather than academics. This has resulted in a few teething and communication problems as Reviewers struggled with teaching commitments and the demands of editing.

The team has, however, enjoyed the experience of working with diverse Authors and has gained many insights which they will apply to their own teaching.

Enjoy the Conference

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AN INTRODUCTION TO THE ALGEBRA OF RANDOM VARIABLES

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Theorems for finding the probability density function of (1) the sum, difference, product and quotient of two independent and continuous random variables, and (2) a function of a single continuous random variable are given and illustrated using simple examples. The theory might stimulate interesting ideas for the Mathematical VCE Methods Unit 4 SAC Analysis Task 2 (Probability).

Introduction

There are many real world problems (particularly in the field of engineering) that require application of algebraic operations to random variables (1, pp 6-9). For example, the sum of independent normal random variables arises in problems where the probability that a given number of people will exceed a specified maximum weight limit is required. While many students might naively assume that the usual rules of algebra apply in such problems, this assumption is not correct.

any pattern and so are not classified, and if this occurs it is included in the feedback as well. Teachers can interview these students to establish their real level of understanding. In this example, the diagnosis is reported in 5 stages (0 to 4). These stages relate only to this topic – they are not yet linked to an external framework such as VELS levels.

Sum of two independent and continuous random variables

Theorem 1

Suppose that X and Y are two independent and continuous random variables with pdf's $f(x)$ and $g(y)$ respectively. Then the sum $U = X + Y$ is a continuous random variable with pdf given by

$$h(u) = \int_{-\infty}^{+\infty} f(u-y)\cdot g(y)\,dy = \int_{-\infty}^{+\infty} g(u-x)\cdot f(x)\,dx \ \ (1, \text{p } 47).$$

Example 1

Let X and Y be two independent standard uniform random variables with pdf's given by $f(x) = \begin{cases} 1 & \text{if } 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$ and $g(y) = \begin{cases} 1 & \text{if } 0 \le y \le 1 \\ 0 & \text{otherwise} \end{cases}$ respectively. Then the sum $U = X + Y$ is a continuous random variable with pdf given by

$$h(u) = \int_{-\infty}^{+\infty} f(u-y)\cdot g(y)\,dy = \int_{0}^{1} f(u-y)\,dy$$

where

$$f(u-y) = \begin{cases} 1 & \text{if } 0 \le u-y \le 1 \Leftrightarrow u-1 \le y \le u \\ 0 & \text{otherwise} \end{cases}$$

Since $0 \le x \le 1$ and $0 \le y \le 1$ it follows that the sample space of $U = X + Y$ is $0 \le u \le 2$. Figure 1 (a) suggests that there are three cases to consider.

Case 1: $0 \le u \le 1 \Rightarrow 0 \le y \le u$. Then $h(u) = \int_0^u 1\,dy = u$.

Case 2: $1 \le u \le 2 \Rightarrow u-1 \le y \le 1$. Then $h(u) = \int_{u-1}^{1} 1\,dy = 2-u$.

Case 3: $u < 0$ or $u > 2 \Rightarrow y < 0$ or $y > 1$. Then $h(u) = 0$.

Therefore

$$h(u) = \begin{cases} u & \text{if } 0 \le u \le 1 \\ 2-u & \text{if } 1 \le u \le 2 \\ 0 & \text{otherwise} \end{cases}$$

This defines a triangular distribution and is shown in Figure 1 (b).

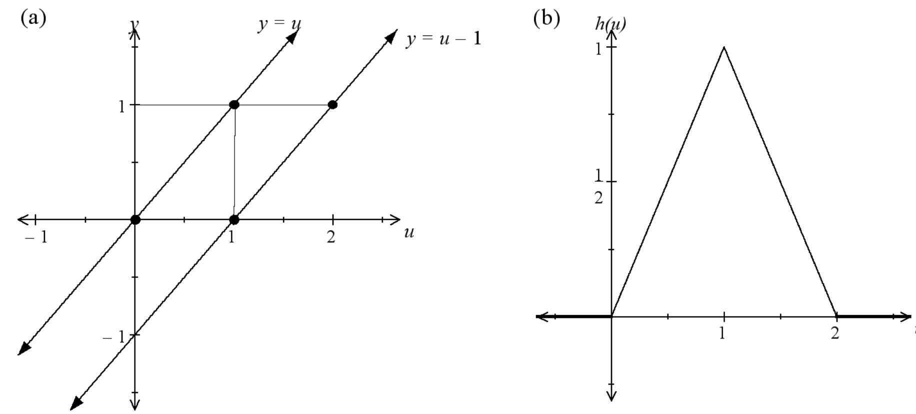

Figure 1. (a) Regions of the uy-plane defined by $u-1 \le y \le u$ *and* $0 \le y \le 1$. *(b) The pdf of* $U = X + Y$.

Example 2

Let X and Y be two independent standard normal random variables with pdf's given by $f(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$ for $-\infty < x < +\infty$ and $g(y) = \frac{1}{\sqrt{2\pi}} e^{-y^2/2}$ for $-\infty < y < +\infty$ respectively (2, p 146). Then the sum $U = X + Y$ is a continuous random variable with the sample space $-\infty < u < +\infty$ and pdf given by:

$$h(u) = \int_{-\infty}^{+\infty} f(u-y)\cdot g(y)\,dy = \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{-(u-y)^2/2} e^{-y^2/2}\,dy$$

$$= \frac{1}{2\pi} e^{-u^2/4} \int_{-\infty}^{+\infty} e^{-\left(y-\frac{u}{2}\right)^2} dy = \frac{1}{2\sqrt{\pi}} e^{-u^2/4}\left(\frac{1}{\sqrt{\pi}}\int_{-\infty}^{+\infty} e^{-\left(y-\frac{u}{2}\right)^2} dy\right).$$

Note: $e^{-(u-y)^2/2} e^{-y^2/2} = e^{-u^2/4} e^{-\left(y-\frac{u}{2}\right)^2}$.

Since $\frac{1}{\sqrt{\pi}} e^{-\left(y-\frac{u}{2}\right)^2}$ is the pdf of a normal random variable ($\mu = \frac{u}{2}$ and $\sigma = \frac{1}{\sqrt{2}}$) (2, p 145) it follows that $\frac{1}{\sqrt{\pi}}\int_{-\infty}^{+\infty} e^{-\left(y-\frac{u}{2}\right)^2}\, dy = 1$. Therefore

$$h(u) = \frac{1}{\sqrt{4\pi}} e^{-u^2/4}$$

and is recognised as a normal distribution with $\mu = 0$ ($= \mu_X + \mu_Y$) and $\sigma^2 = 2$ ($= \sigma_X^2 + \sigma_Y^2$) (2, p 145).

In general, the sum of two independent normal random variables with means and variances μ_1 and σ_1^2 and μ_2 and σ_2^2 respectively is a normal random variable with mean $\mu_1 + \mu_2$ and variance $\sigma_1^2 + \sigma_2^2$ (2, p 146). It follows that the sum of *n* independent normal random variables is a normal random variable with mean $\mu_1 + \mu_2 + \ldots + \mu_n$ and variance $\sigma_1^2 + \sigma_2^2 + \ldots + \sigma_n^2$.

Difference of two independent and continuous random variables

Theorem 2

Suppose that X and Y are two independent and continuous random variables with pdf's *f(x)* and *g(y)* respectively. Then the difference $U = X - Y$ is a continuous random variable with pdf given by

$$h(u) = \int_{-\infty}^{+\infty} f(u+y)\cdot g(y)\, dy = \int_{-\infty}^{+\infty} g(u+x)\cdot f(x)\, dx \text{ (1, p 48).}$$

Example 3

Let X and Y be two independent and identically distributed exponential random variables with pdf's given by $f(x) = \begin{cases} \lambda e^{-\lambda x} & \text{if } x \ge 0 \\ 0 & \text{otherwise} \end{cases}$ and $g(y) = \begin{cases} \lambda e^{-\lambda y} & \text{if } y \ge 0 \\ 0 & \text{otherwise} \end{cases}$ respectively (2, p 78). Then the difference $U = X - Y$ is a continuous random variable with pdf given by

$$h(u) = \int_{-\infty}^{+\infty} f(u+y)\cdot g(y)\, dy = \int_{0}^{+\infty} f(u+y)\lambda e^{-\lambda y}\, dy$$

where

$$f(u+y) = \begin{cases} \lambda e^{-\lambda(u+y)} & \text{if } u+y \ge 0 \Leftrightarrow y \ge -u \\ 0 & \text{otherwise} \end{cases}$$

Since $x \ge 0$ and $y \ge 0$ it follows that the sample space of $U = X - Y$ is $-\infty < u < +\infty$. There are two cases to consider.

Case 1: $u < 0$. Then $h(u) = \int_{-u}^{+\infty} \lambda e^{-\lambda(u+y)} \lambda e^{-\lambda y}\, dy = \lambda^2 e^{-\lambda u} \int_{-u}^{+\infty} e^{-2\lambda y}\, dy = \frac{\lambda}{2} e^{\lambda u}$.

Case 2: $u \ge 0$. Then $h(u) = \int_{0}^{+\infty} \lambda e^{-\lambda(u+y)} \lambda e^{-\lambda y}\, dy = \lambda^2 e^{-\lambda u} \int_{0}^{+\infty} e^{-2\lambda y}\, dy = \frac{\lambda}{2} e^{-\lambda u}$.

Therefore

$$h(u) = \begin{cases} \frac{\lambda}{2} e^{-\lambda u} & \text{if } u \ge 0 \\ \frac{\lambda}{2} e^{\lambda u} & \text{if } u < 0 \end{cases}$$

Product of two independent and continuous random variables

Theorem 3

Suppose that X and Y are two independent and continuous random variables with pdf's *f(x)* and *g(y)* respectively and $\Pr(X = 0) = \Pr(Y = 0) = 0$. Then the product $U = XY$ is a continuous random variable with pdf given by

$$h(u) = \int_{-\infty}^{+\infty} \frac{1}{|y|} f\left(\frac{u}{y}\right)\cdot g(y)\, dy = \int_{-\infty}^{+\infty} \frac{1}{|x|} f(x)\cdot g\left(\frac{u}{x}\right) dx \text{ (1, p 91).}$$

An algorithm for implementing this result, along with an implementation of the algorithm in a CAS, can be found in Reference 4.

Example 4

Let X and Y be two independent standard uniform random variables with pdf's given by $f(x) = \begin{cases} 1 & \text{if } 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$ and $g(y) = \begin{cases} 1 & \text{if } 0 \le y \le 1 \\ 0 & \text{otherwise} \end{cases}$ respectively. Then the product

$U = XY$ is a continuous random variable with pdf given by

$$h(u) = \int_{-\infty}^{+\infty} \frac{1}{|y|} f\left(\frac{u}{y}\right) \cdot g(y)\, dy = \int_{0}^{1} \frac{1}{y} f\left(\frac{u}{y}\right) dy$$

where

$$f\left(\frac{u}{y}\right) = \begin{cases} 1 & \text{if } 0 \le \frac{u}{y} \le 1 \Leftrightarrow u \le y \le 1 \\ 0 & \text{otherwise} \end{cases}$$

Since $0 \le x \le 1$ and $0 \le y \le 1$ it follows that the sample space of $U = XY$ is $0 \le u \le 1$ (from which it follows that $0 \le \frac{u}{y} \le 1 \Leftrightarrow 1 \le \frac{y}{u} < +\infty \Leftrightarrow u \le y < +\infty \Leftrightarrow u \le y \le 1$). There are two cases to consider.

Case 1: $0 < u \le 1$. Then $h(u) = \int_{u}^{1} \frac{1}{y}\, dy = -\ln(u)$.

Case 2: $u \le 0$ or $u > 1$. Then $h(u) = 0$.

Therefore

$$h(u) = \begin{cases} -\ln(u) & \text{if } 0 < u \le 1 \\ 0 & \text{otherwise} \end{cases}$$

Example 5

Let *X* and *Y* be two independent Cauchy random variables each with zero median and therefore pdf's given by $f(x) = \frac{1}{\pi}\frac{a}{(a^2 + x^2)}$ for $-\infty < x < +\infty$ and $g(y) = \frac{1}{\pi}\frac{b}{(b^2 + y^2)}$ for $-\infty < y < +\infty$ respectively (2, p 48). Then the product $U = XY$ is a continuous random variable with sample space $-\infty < u < +\infty$ and pdf given by:

$$h(u) = \int_{-\infty}^{+\infty} \frac{1}{|y|} f\left(\frac{u}{y}\right) \cdot g(y)\, dy = \frac{ab}{\pi^2} \int_{-\infty}^{+\infty} \frac{1}{|y|} \frac{1}{\left(a^2 + \frac{u^2}{y^2}\right)} \cdot \frac{1}{(b^2 + y^2)}\, dy$$

$$= \frac{2ab}{\pi^2} \int_{0}^{+\infty} \frac{y}{(a^2 y^2 + u^2)(b^2 + y^2)}\, dy \quad \text{(using symmetry)}$$

$$= \frac{2ab}{\pi^2 (u^2 - a^2 b^2)} \int_{0}^{+\infty} \frac{y}{b^2 + y^2} - \frac{a^2 y}{a^2 y^2 + u^2}\, dy \quad \text{(using partial fraction decomposition)}$$

$$= \frac{2ab}{\pi^2 (u^2 - a^2 b^2)} \lim_{\alpha \to +\infty} \int_{0}^{\alpha} \frac{y}{b^2 + y^2} - \frac{a^2 y}{a^2 y^2 + u^2}\, dy \quad \text{(since the integral is improper)}$$

$$= \frac{ab}{\pi^2 (u^2 - a^2 b^2)} \lim_{\alpha \to +\infty} \left[\ln\left(\frac{b^2 + y^2}{a^2 y^2 + u^2} \right) \right]_0^{\alpha}$$

$$= \frac{ab}{\pi^2 (u^2 - a^2 b^2)} \left(\ln\left(\frac{1}{a^2}\right) - \ln\left(\frac{b^2}{u^2}\right) \right) = \frac{ab}{\pi^2 (u^2 - a^2 b^2)} \ln\left(\frac{u^2}{a^2 b^2}\right).$$

This result is in agreement with the special cases $a = b = 1$ (4, p 304) and $b = a$ (1, p 158 Question 4.5). Note that $h(u)$ is undefined for $u = 0$ and indeterminant for $u = \pm ab$. However,

$$\lim_{u \to \pm ab} \left[\frac{ab}{\pi^2 (u^2 - a^2 b^2)} \ln\left(\frac{u^2}{a^2 b^2}\right) \right] = \frac{1}{ab\pi^2} \lim_{t \to \pm 1} \frac{\ln(t^2)}{t^2 - 1} = \frac{1}{ab\pi^2}$$

making the substitution $u = abt$ and using either l'Hôpital's Rule or a CAS. Furthermore, it follows that the function

$$H(u) = \int_{-\infty}^{u} h(w)\, dw = \int_{-\infty}^{u} \frac{ab}{\pi^2 (w^2 - a^2 b^2)} \ln\left(\frac{w^2}{a^2 b^2}\right) dw = \frac{1}{\pi^2} \int_{-\infty}^{u/(ab)} \frac{\ln(t^2)}{t^2 - 1}\, dt$$

(where the substitution $w = abt$ has been made) is continuous over $-\infty < u < +\infty$. In particular, $H(0) = \frac{1}{2}$ (as expected since the pdf is an even function), $H(-ab) = \frac{1}{4}$ and $H(ab) = \frac{3}{4}$. This defines the cumulative distribution function (cdf) $\Pr(U \le u)$ of *U*.

There is no requirement that the pdf of a random variable be continuous over its sample space (5, p 29).

Quotient of two independent and continuous random variables

Theorem 4

Suppose that X and Y are two independent and continuous random variables with pdf's $f(x)$ and $g(y)$ respectively and $\Pr(X = 0) = \Pr(Y = 0) = 0$. Then the quotient $U = \frac{X}{Y}$ is a continuous random variable with pdf given by

$$h(u) = \int_{-\infty}^{+\infty} |\, y \,|\, f(uy) \cdot g(y)\, dy = \int_{-\infty}^{+\infty} |\, x \,|\, f(x) \cdot g(ux)\, dx \quad (1, \text{p } 91).$$

Example 6

Let X and Y be two independent standard uniform random variables with pdf $f(x)$ and $g(y)$ respectively. Then the quotient $U = \frac{X}{Y}$ is a continuous random variable with pdf given by

$$h(u) = \int_{-\infty}^{+\infty} |\, y \,|\, f(uy) \cdot g(y)\, dy = \int_{0}^{1} y f(uy)\, dy$$

where

$$f(uy) = \begin{cases} 1 & \text{if } 0 \le uy \le 1 \Leftrightarrow 0 \le y \le \frac{1}{u} \\ 0 & \text{otherwise} \end{cases}$$

Since $0 \le x \le 1$ and $0 \le y \le 1$ it follows that the sample space of $U = \frac{X}{Y}$ is $0 \le u < +\infty$. Figure 2 (a) suggests that there are three cases to consider.

Case 1: $0 \le u \le 1 \Rightarrow 0 \le y \le 1$. Then $h(u) = \int_0^1 y\, dy = \frac{1}{2}$.

Case 2: $1 \le u < +\infty \Rightarrow 0 \le y \le \frac{1}{u}$. Then $h(u) = \int_0^{1/u} y\, dy = \frac{1}{2u^2}$.

Case 3: $u < 0 \Rightarrow y < 0$. Then $h(u) = 0$.

Therefore

$$h(u) = \begin{cases} \frac{1}{2} & \text{if } 0 \le u \le 1 \\ \frac{1}{2u^2} & \text{if } 1 \le u < +\infty \\ 0 & \text{otherwise} \end{cases}$$

This pdf is shown in Figure 2 (b).

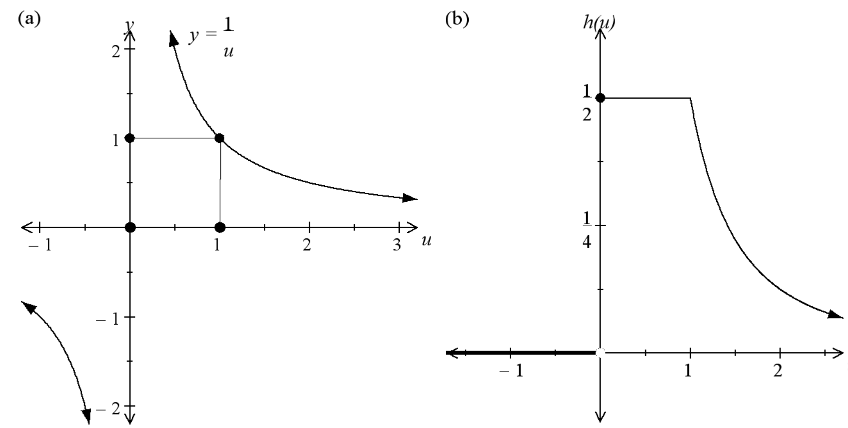

Figure 2. (a) Regions of the uy-plane defined by $0 \le y \le 1/u$ *and* $0 \le y \le 1$*. (b) The pdf of* $U = X/Y$*.*

Example 7

Let X and Y be two independent normal random variables each with zero mean and therefore pdf's given by

$$f(x) = \frac{1}{\sigma_X \sqrt{2\pi}} e^{\frac{-x^2}{2\sigma_X^2}} \quad \text{for } -\infty < x < +\infty$$

$$g(y) = \frac{1}{\sigma_Y \sqrt{2\pi}} e^{\frac{-y^2}{2\sigma_Y^2}} \quad \text{for } -\infty < y < +\infty$$

respectively (2, p 145). Then the quotient $U = \dfrac{X}{Y}$ is a continuous random variable with sample space $-\infty < u < +\infty$ and pdf given by:

$$h(u) = \int_{-\infty}^{+\infty} |\, y \,|\, f(uy) \cdot g(y)\, dy$$

$$= \frac{1}{2\pi\sigma_X\sigma_Y} \int_{-\infty}^{+\infty} |\, y \,|\, e^{\frac{-u^2y^2}{2\sigma_X^2}} e^{\frac{-y^2}{2\sigma_Y^2}}\, dy$$

$$= \frac{1}{\pi\sigma_X\sigma_Y} \int_{0}^{+\infty} y e^{\frac{-y^2}{2}\left(\frac{u^2}{\sigma_X^2} + \frac{1}{\sigma_Y^2}\right)}\, dy$$

$$= \frac{1}{\pi\sigma_X\sigma_Y} \int_{0}^{+\infty} y e^{\frac{-ay^2}{2}}\, dy$$

where $a = \dfrac{u^2}{\sigma_X^2} + \dfrac{1}{\sigma_Y^2} = \dfrac{u^2\sigma_Y^2 + \sigma_X^2}{\sigma_X^2\sigma_Y^2}$. However,

$$\int_{0}^{+\infty} y e^{\frac{-ay^2}{2}}\, dy = \lim_{\alpha \to +\infty} \int_{0}^{\alpha} y e^{\frac{-ay^2}{2}}\, dy = \lim_{\alpha \to +\infty} \left[-\frac{1}{a} e^{\frac{-ay^2}{2}} \right]_0^{\alpha} = \lim_{\alpha \to +\infty} \left(\frac{1}{a} - \frac{1}{a} e^{\frac{-a\alpha^2}{2}} \right) = \frac{1}{a}$$

and so

$$h(u) = \frac{1}{\pi\sigma_X\sigma_Y} \frac{1}{a} = \frac{1}{\pi} \frac{\sigma_X\sigma_Y}{(u^2\sigma_Y^2 + \sigma_X^2)} = \frac{1}{\pi} \frac{\beta}{(u^2 + \beta^2)} \text{ where } \beta = \frac{\sigma_X}{\sigma_Y}.$$

This is recognised as the pdf of a random variable that follows a Cauchy distribution with median equal to zero and shape factor β (2, pp 48, 149).

By symmetry it follows that the random variable $\dfrac{1}{U} = \dfrac{Y}{X}$ also follows a Cauchy distribution with median equal to zero but with shape factor $\dfrac{1}{\beta}$. Letting $b = \dfrac{1}{a}$ in Example 5 it therefore follows that the pdf of the quotient of two independent and identically distributed Cauchy random variables with zero median is $\dfrac{1}{\pi^2} \dfrac{\ln(u^2)}{(u^2 - 1)}$:

If $X \sim \text{Cauchy}\,(0, a)$ and $Y \sim \text{Cauchy}\left(0, \dfrac{1}{a}\right)$ then $\dfrac{1}{Y} = V \sim \text{Cauchy}\,(0, a)$. Therefore $\dfrac{X}{V} = XY$ and so its pdf is $\dfrac{1}{\pi^2} \dfrac{\ln(u^2)}{(u^2 - 1)}$.

This result can be confirmed by directly applying Theorem 4. The commonly cited result given by Springer (1, p 158 Question 4.6) is therefore incorrect.

Functions of a single continuous random variable

Suppose that X is a continuous random variable with a known pdf and let k be a function defined over the sample space of X. There are three common methods for finding the pdf of the random variable defined by $U = k(X)$: the method of distribution functions, the method of transformations and the method of moment generating functions.

The method of distribution functions

The method of distribution functions involves finding the cumulative distribution function (cdf) of U and then differentiating it to get the pdf of U.

Example 8

Suppose that X is a random variable with pdf $f(x)$ and let $U = X^2$. Then the cdf of U is given by:

$$H(u) = \Pr(U \le u) = \Pr\left(X^2 \le u\right) = \Pr\left(-\sqrt{u} \le X \le \sqrt{u}\right) = \int_{-\sqrt{u}}^{\sqrt{u}} f(x)\, dx.$$

Then the pdf of U is given by:

$$h(u) = \frac{dH}{du} = \frac{d}{du}\left(\int_{-\sqrt{u}}^{\sqrt{u}} f(x)\, dx \right) = \frac{1}{2\sqrt{u}} \left(f\left(\sqrt{u}\right) + f\left(-\sqrt{u}\right) \right)$$

using the chain rule and the Fundamental Theorem of Calculus:

$$\frac{d}{du}\left(\int_{-\sqrt{u}}^{\sqrt{u}} f(x)\,dx\right) = \frac{d}{du}\left(\int_{-\sqrt{u}}^{0} f(x)\,dx + \int_{0}^{\sqrt{u}} f(x)\,dx\right) = \frac{d}{du}\left(-\int_{0}^{-\sqrt{u}} f(x)\,dx + \int_{0}^{\sqrt{u}} f(x)\,dx\right)$$

To find $\frac{d}{du}\left(-\int_{0}^{-\sqrt{u}} f(x)\,dx\right)$, substitute $w = -\sqrt{u}$ and use the chain rule:

$\frac{d}{du}\left(-\int_{0}^{-\sqrt{u}} f(x)\,dx\right) = -\frac{d}{dw}\left(\int_{0}^{w} f(x)\,dx\right) \times \frac{dw}{du} \quad = -f(w) \times \left(\frac{-1}{2\sqrt{u}}\right)$ since, from the Fundamental Theorem of Calculus, $\frac{d}{dw}\left(\int_{0}^{w} f(x)\,dx\right) = f(w)$. Therefore $\frac{d}{du}\left(-\int_{0}^{-\sqrt{u}} f(x)\,dx\right) = \frac{1}{2\sqrt{u}} f\left(-\sqrt{u}\right)$. Similarly, $\frac{d}{du}\left(\int_{0}^{\sqrt{u}} f(x)\,dx\right) = \frac{1}{2\sqrt{u}} f\left(\sqrt{u}\right)$.

If X is a continuous standard uniform random variable then the sample space of $U = X^2$ is $0 \le u \le 1$ and $h(u) = \frac{1}{2\sqrt{u}}$ for $0 < u \le 1$ and zero otherwise.

If X is a standard normal random variable then the sample space of $U = X^2$ is $0 \le u < +\infty$ and $h(u) = \frac{1}{\sqrt{2\pi}}\frac{1}{\sqrt{u}} e^{-u/2}$ for $0 < u < +\infty$ and zero otherwise (this is recognised as the pdf of a random variable that follows a chi-squared distribution with 1 degree of freedom (2, pp 52, 148)).

Probability integral transform theorem

Suppose that X is a continuous random variable with continuous cdf $F(x)$. Then $U = F(X)$ is a continuous standard uniform random variable (6, pp 242-243).

For example, if X is an exponential random variable with pdf $f(x) = \lambda e^{-\lambda x}$ and therefore cdf $F(x) = 1 - e^{-\lambda x}$, then $U = 1 - e^{-\lambda X}$ is a continuous standard uniform random variable.

The probability integral transform theorem can be re-stated as follows: Suppose that X is a continuous random variable with pdf $f(x)$ and continuous cdf $F(x)$. Suppose that Y is a continuous standard uniform random variable. Then $U = F^{-1}(Y)$ is a random variable with the same probability distribution as X (6, p 244).

Proof: The cdf of U is given by:

$$H(u) = \Pr(U \le u) = \Pr\left(F^{-1}(Y) \le u\right) = \Pr\left(Y \le F(u)\right) = \int_{0}^{F(u)} 1\,dx = F(u).$$

Therefore the pdf of U is given by $h(u) = \frac{dH}{du} = \frac{dF}{du} = f(u)$ and so U has the same probability distribution as X.

For example, if X is an exponential random variable with pdf $f(x) = \lambda e^{-\lambda x}$ and therefore cdf $F(x) = 1 - e^{-\lambda x} \Rightarrow F^{-1}(x) = -\frac{1}{\lambda}\ln(1-x)$ and Y is a continuous standard uniform random variable, then $U = F^{-1}(Y) = -\frac{1}{\lambda}\ln(1-Y)$ is an exponential random variable (with pdf $g(u) = \lambda e^{-\lambda u}$). In fact, it is easy to show that $1-Y$ is also a standard uniform random variable and so it follows that $-\frac{1}{\lambda}\ln(Y)$ is also an exponential random variable.

It follows from the probability integral transform theorem that a random sample $S = \{y_1, y_2, \ldots y_N\}$ from a standard uniform distribution can be used to generate a random sample from another probability distribution with continuous cdf F by using the transformation $F^{-1}(y)$ where $y \in S$ (provided that a unique inverse function F^{-1} exists). This is known as the inverse transform sampling method.

The method of transformations

Theorem 5

Suppose that X is a continuous random variable with pdf $f(x)$ and sample space D. If the function $u = k(x)$ is differentiable in D and either $\frac{dk}{dx} > 0$ or $\frac{dk}{dx} < 0$ in D then $U = k(X)$ is a continuous random variable with pdf given by $f\left(k^{-1}(u)\right)\left|\frac{dk^{-1}(u)}{du}\right|$ over its sample space (7, p 267). Special result: The pdf of the random variable $U = aX + b$ over its sample space is $\frac{1}{|a|} f\left(\frac{u-b}{a}\right)$.

Example 9

Suppose that X is a standard uniform random variable with pdf *f(x)* and let $U = e^X$. The function $k(x) = e^x$ is differentiable for all x and $\frac{dk}{dx} > 0$ for all x. Then $U = e^X$ is a continuous random variable with sample space $1 \le u \le e$. $k(x) = e^x \Rightarrow k^{-1}(x) = \ln(x)$ and so

$$f\left(k^{-1}(u)\right)\left|\frac{dk^{-1}(u)}{du}\right| = 1 \cdot \left|\frac{1}{u}\right| = \frac{1}{u}$$

Note: $f\left(k^{-1}(u)\right) = \begin{cases} 1 & \text{if } 0 \le k^{-1}(u) \le 1 \Leftrightarrow 1 \le u \le e \\ 0 & \text{otherwise} \end{cases}$.

Therefore the pdf of U is

$$h(u) = \begin{cases} \frac{1}{u} & \text{if } 1 \le u \le e \\ 0 & \text{otherwise} \end{cases}$$

The method of moment-generating functions

Definition

The moment-generating function (mgf) of a continuous random variable X is defined to be $m_X(t) = E\left(e^{tX}\right)$ (7, p 169).

Theorem 6

Suppose that $U = k(X)$ is a single-valued function of a random variable X with pdf *f(x)*. Then the mgf of U is $m_U(t) = E\left(e^{tk(X)}\right) = \int_{-\infty}^{+\infty} e^{tk(x)} f(x)\, dx$ (7, p 172). Special result: If $U = aX + b$ then $m_U(t) = e^{bt} m_X(at)$.

Theorem 7 (uniqueness theorem)

Suppose that moment-generating functions exist for the two random variables X and U and are given by $m_X(t)$ and $m_U(t)$ respectively. If $m_U(t) = m_X(t)$ for all values of t then U and X have the same probability distribution (7, pp 271-272).

Example 10

Suppose that X is a standard normal random variable and let $U = X^2$. Then by Theorem 6 the mgf of U is given by:

$$m_U(t) = E\left(e^{tX^2}\right) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{tx^2} e^{-x^2/2}\, dx = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-x^2(1-2t)/2}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-\frac{x^2}{2b}}\, dx = \sqrt{b}\left(\frac{1}{\sqrt{b}\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-\frac{x^2}{2b}}\, dx\right)$$

where $b = \frac{1}{1-2t}$. Since $\frac{1}{\sqrt{b}\sqrt{2\pi}} e^{\frac{-x^2}{2b}}$ is the pdf of a normal random variable ($\mu = 0$ and $\sigma = \sqrt{b}$) (3. p 145) it follows that $\frac{1}{\sqrt{b}\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{\frac{-x^2}{2b}}\, dx = 1$. Therefore $m_U(t) = \sqrt{b} = \frac{1}{(1-2t)^{1/2}}$.

Comparing this result with a table of moment-generating functions for common continuous random variables (see for example Reference 7), $m_U(t)$ is recognised as the mgf of a random variable that follows a chi-squared distribution with 1 degree of freedom. It follows from theorem 7 that the pdf of U is

$$h(u) = \frac{1}{\sqrt{2\pi}} \frac{1}{\sqrt{u}} e^{-u/2} \text{ for } 0 < u < +\infty \text{ and zero otherwise.}$$

Theorem 8

Let X_1, X_2, ... X_n be independent random variables with mgf's $m_{X_1}(t)$, $m_{X_2}(t)$, ... $m_{X_n}(t)$ respectively. If $U = X_1 + X_2 + \ldots + X_n$ then $m_U(t) = m_{X_1}(t) \cdot m_{X_2}(t) \cdot \ldots \cdot m_{X_n}(t)$ (7, p 273).

References

1. Springer M. D. (1979). *The Algebra of Random Variables.* New York: John Wiley and Sons.
2. Evans M., Hastings N. and Peacock B. (2000). *Statistical Distributions.* New York: John Wiley and Sons.
3. Glen A. G., Leemis L. M. and Drew J. H. (2004). Computing the Distribution of the Product of Two Continuous Random Variables. *Computational Statistics & Data Analysis, 44 (3),* pp 451-464.
4. Ride P. R. (1965). Distributions of Product and Quotient of Cauchy Variables. *The American Mathematical Monthly, 72 (3),* pp 303-305.
5. Romano, J. P. and Siegel A. F. (1986). *Counterexamples in Probability and Statistics.* Boca Raton: Chapman and Hall/CRC.
6. Roussas G. G. (1997). *A Course in Mathematical Statistics.* San Diego: Academic Press.
7. Mendenhall W., Scheaffer R. L. and Wackerley D. D. (1996). *Mathematical Statistics With Applications.* Boston: Duxbury Press.

WORKING MATHEMATICALLY IN A RURAL CONTEXT

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In 2008 we jointly coordinated a project, the aim of which was to bring the mathematics of farming into the classroom. Activities developed for secondary schools as a result of this project have been edited, illustrated and made available on a public website. The purpose of this paper is to outline these activities and suggest how they may be used in the classroom.

The project

The project called *Working Mathematically* in a Rural Context (commonly known as *WMRural*) involved 18 participating teachers from 6 small schools in the Riverina of NSW, four local farmers, and ourselves. We met for an initial workshop in Wagga Wagga in March and on three further occasions during the year by videoconference. The farmers acted as advisors throughout the project. Each school developed three Working Mathematically activities focused on rural life, working with a farmer to adapt authentic rural activities to the classroom. They then trialed their activities with one or more classes in their school and revised them. The farmers also visited each school and talked to students about the various ways in which they used mathematics on the farm. A private web site was set up to facilitate communication between the teachers, farmers and ourselves.

Teachers' formal and informal evaluations indicated that the use of authentic activities set in a familiar context led to far greater engagement on the part of their students, especially the boys. Students who normally did not participate well in mathematics lessons began to show an interest and some of them even showed surprisingly deep understanding of various mathematical aspects of life on the farm. In short, mathematics began to make sense.

In 2009, we revised, illustrated and extended the activities following teachers' and farmers' suggestions, and edited them to a common format. We also produced an introductory video *Mathematics on the Farm,* in which the four participating farmers recorded how they used mathematics on a day-to-day basis. All these materials are now freely available at http://www.wmrural.net for anyone to download.

The activities

There are currently 19 activities on the WMRural website, grouped under the three headings: Cropping industries, Animal industries, and General. Although the activities are typical of problems arising in rural contexts, several (e.g., an activity about rainwater tanks) would be of interest to students anywhere.

To help teachers decide whether an activity is suitable for their class, a table on the web site indicates firstly, which strands of the syllabus are covered and secondly, which year groups (between year 7 and year 11) they are most suitable for. The student worksheet accompanying each activity is divided into sections. The early sections usually contain relatively straightforward calculations and the later sections are more complex extensions suitable for more advanced students. The various possibilities are explained in the teachers' notes, which also indicate the specific syllabus outcomes addressed.

We found that many mathematics teachers, especially those who have only recently moved to the country, are unfamiliar with the details of operating a farm. A single page background information sheet is therefore provided to accompany each activity, including links to sources of relevant, up-to-date information on the internet. There are also illustrative photographs which can be displayed on a whiteboard, and of course there are full answers to all the student worksheets.

Using the activities

During the project, we found that a number of matters need to be kept in mind before using a WMRural activity in your classroom:

- It is important to plan ahead: Select activities which match your teaching program, decide whether to use the activity to introduce one or more new ideas or to consolidate topics already covered and consider whether the activity may be better as class work or as an independent assignment.
- Select suitable activities: Select activities where the context is likely to be familiar and/or of interest to your students. If the topic is unfamiliar, take care to fill in the background (e.g., by using the background information sheet) well before starting the activity.

- Tailor lessons to suit your class: Take students' ability into account by selecting the most suitable parts of the activity. As mentioned previously, the easier questions are usually found in early sections of the activity. Also, within each section, the questions are sequenced to provide scaffolding. By omitting one or more of the leading questions, you can leave it to your more capable students to decide upon the route to the final solution.
- Allow cognitive conflict to develop: Encourage students to check theoretical calculations against reality and to explain and learn from any inconsistencies. For example, if you have to fit posts 2 m apart along a 12 m fence, are there 12 ÷ 2 posts along each side? If not, why not? How *do* you find the number of posts?
- Use physical demonstrations and manipulatives wherever possible. The dam activity is a good example as the analysis would be difficult if you only had drawings to use. Another example, in the activity concerning the spreading of fertiliser, is to use a backyard lawn spreader to illustrate the relation between tractor speed and rate of fertiliser application.
- Bring an authority into your classroom: Ask him or her to talk about the broader background to a particular activity. If you cannot find a farmer, then try local suppliers or the agriculture teacher, or ask among your students' parents. If all else fails, use the video *Mathematics on the Farm* from the web site.

Further ideas

The WMRural web site has aroused interest all over the world. (We suspect that many users are expatriate Australians bringing an Australian flavour to their lessons!) We are committed to keeping the site alive for some years to come, and would welcome suggestions for further activities or ideas on using the site more effectively.

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STUDENTS' PERCEPTIONS OF MATHEMATICS TEACHING AND LEARNING DURING THE MIDDLE YEARS

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This paper will explore some of the findings from a longitudinal study of engagement in middle years school mathematics. A group of 20 students participated in individual interviews, focus group discussions and classroom observations over the course of their transition from Primary to Secondary school in New South Wales. The students identified teachers they perceived to be 'good' mathematics teachers. Pedagogies that maintained engagement in mathematics were explored along with students' desired attributes of 'good' mathematics teachers.

Introduction

Engagement in mathematics during the middle years of schooling (Years 5 to 8 in New South Wales) has been of some concern in Australia in recent decades. The National Numeracy Review (Commonwealth of Australia, 2008) reported that although the levels of mathematics achievement are quite good when compared to international standards, there is an unacceptable number of Australian students who do not achieve appropriate levels of proficiency. The report claims many students fail to enjoy or see the personal relevance of mathematics and few voluntarily continue its study. Although arguably attitudes change throughout the school years, once formed, negative attitudes towards mathematics are difficult to change and can persist into adult life (Newstead, 1998). These findings have been reflected in research both in Australia (State of Victoria Department of Education and

Training, 2004; Sullivan, McDonough, & Harrison, 2004) and internationally (Anderman & Midgley, 1997; Leckey, 2000).

As part of a qualitative longitudinal study on engagement in mathematics during the middle years, a group of twenty students were asked to provide their views on mathematics teaching and learning through individual interviews, focus group discussions and classroom observations. Data was collected over the course of their transition from the final year of primary schooling through to their second year of secondary education. The students were asked to discuss aspects of their favourite mathematics lessons and the qualities of a good mathematics teacher. This paper will explore their perspectives of mathematics and the pedagogies that they found engaging, as well as their views on 'good' mathematics teaching.

First, a broad overview of previous research on good teaching will be provided. Next, a brief description of the research design will be provided, followed by an exploration of the participants' views of attributes of a 'good' teacher. This will be followed by a discussion of the students' perceptions of their learning experiences in mathematics over the course of the study.

Previous Research Findings

Arguably one of the most significant factors impacting on engagement in mathematics is teaching practices, or pedagogy (Hayes, Mills, Christie, & Lingard, 2006; NSW Department of Education and Training, 2003). In summary, research has found that aspects of teaching that have a strong influence on student engagement in mathematics are: a strong pedagogical content knowledge; relevant mathematics that is connected to the lives of the learners; incorporation of higher-order thinking skills; a supportive and inclusive classroom environment; and learning and teaching activities that lead to 'deep' understanding of mathematical concepts (Askew, Brown, Rhodes, Johnson, & Wiliam, 1997; Australian Association of Mathematics Teachers [AAMT], 2006; Commonwealth of Australia, 2008) Thus pedagogy that is ineffective alongside negative school experiences can potentially lead to poor or negative motivation, leading to lowered levels of engagement in mathematics, whereas effective pedagogy leads to positive motivation, leading to higher quality levels of engagement.

Significant research into quality teaching in general (Haberman, 2005; Lingard et al., 2001), and specific to mathematics (Askew et al., 1997) has resulted in the construction of several 'frameworks' that describe what quality teaching and learning should look like in the classroom. However, such frameworks such as the Productive Pedagogies (Lingard et al., 2001) and the NSW Quality Teaching (NSW Department of Education and Training, 2003) frameworks are not without limitations. Hayes et al. (2006) criticised the research that formed the Productive Pedagogies framework due to an 'absence of student voice' and ponder that the generational gap between teachers and students today may be wider than ever before, making it more of a challenge to fully appreciate the effects of pedagogy on today's students.

There is a gap in the research when it comes to recognising students' voices and their perceptions of quality teaching and learning in mathematics in Australia. It can be argued that middle years students' perceptions are naïve due to their age and stage of maturity. However, if motivation and engagement are an individual's response to his or her experiences and environment, then it follows that students' voices should be heard and should help to inform frameworks that support quality teaching and learning. The following is a brief description of the research design followed by an exploration of students' perceptions of mathematics teaching and learning.

Research Design

The study was carried out at two sites. The initial phase took place during the students' final year of primary school in a Western Sydney school. The school had been selected as an appropriate site for the study to begin because it was one in which a large proportion of students gained high achievement levels in the Year 5 Basic Skills Numeracy Test in 2007. A 'high achieving' school was chosen due to repeated studies showing moderate to strong correlations between academic achievement and academic self-concept (Barker, Dowson, & McInerney, 2005), and the reasoning that those students who experience positive academic self-concept in mathematics are more likely to be engaged. During the second phase of data collection the students attended the second site, a secondary school within the same area of Western Sydney.

In order to identify prospective participants, the Year 6 cohort of 55 students completed the Motivation and Engagement Scale (High School) with all questions specific to mathematics (Martin, 2008). Twenty students, all of whose results showed strong levels of engagement towards mathematics and intended on attending the same high school, were invited and became participants. The participants represented a diverse range of mathematical abilities, cultural backgrounds, and most came from families with two working parents.

Participants took part in individual interviews at the beginning of the study and again

at the conclusion. They also took part in focus group discussions, which took place once during Year 6, three times during Year 7 and once during Year 8. Other data were collected through two series of classroom observations of two teachers identified by students as 'good' mathematics teachers, and interviews with each of those teachers.

Due to the longitudinal nature of this study, the students experienced two very different school settings that arguably had a significant effect on the outcomes of this study. A brief description of the contexts of each of the schools will now be provided.

The classroom contexts

In order to fully appreciate the perceptions of the participants in this study it is important to have an understanding of the differences between the primary and secondary classroom and school contexts. During their primary schooling, the participants experienced a 'whole school' approach to teaching and learning that incorporated a foundation of cooperative learning and a Multiple Intelligence (Gardner, 1993) approach to curriculum delivery. The students worked in open classrooms where co-teaching occurred and peer collaboration was expected.

When the students entered secondary school, the school was in its third year of operation and considered itself a 'groundbreaking' learning community in which an interdisciplinary approach to learning via an integrated curriculum was delivered. Each student at the school was equipped with a laptop computer and teachers were known as 'learning advisors'. Co-teaching occurred in large, purpose-built learning spaces with each learning advisor taking a role in the facilitation of the group. Mathematics lessons were delivered by a team of four teachers who rotated through four groups of students. This meant the students did not experience the same teacher for two consecutive mathematics lessons. The main teaching and learning resource for mathematics lessons was the students' laptop computers.

During the first term of Year 8, the secondary school made a substantial change in its curriculum structure and reverted back to a more traditional approach, where subject areas were no longer integrated, there was less focus on using laptops in the mathematics classrooms and each group of students had one teacher for the entire year.

The data informing this paper was derived from the student interviews and focus group discussions during their transition from primary to secondary school. The students' perceptions of a 'good' mathematics teacher will now be explored.

What makes a 'good' mathematics teacher?

Whilst in Year 6, the students had already formulated strong opinions on the desired attributes of a 'good' mathematics teacher. They were strongly focused on the following points. A 'good' mathematics teacher:

- is passionate about mathematics;
- 'knows' children;
- explains things well;
- provides assistance by scaffolding rather than providing answers;
- encourages the students to have positive attitudes towards mathematics; and
- acknowledges each students' prior knowledge.

The qualities of a 'good' mathematics teacher as identified by the students strongly reflect several of the attributes described in the AAMT Standards (2006) and those described earlier by Askew et al. (1997). The teachers whom the students had identified as being the 'best' teachers of mathematics over the course of the study had similar attributes to those listed above. In Year 6, the 'best' teacher of mathematics was described by one of the girls:

> "She just puts a lot of enthusiasm in maths and makes it really fun for us. She gets all these different maths activities. She just makes it really fun for us and I quite enjoy maths now because of that."

As the students made the transition to secondary school and experienced a significant number of changes in the organisation of schooling and the pedagogies of several different mathematics teachers, their desired teacher attributes remained similar although there was a much stronger focus on good teacher explanations. The following quote from a Year 7 boy sums up the feelings of many of the participants:

> "I think a good maths teacher is Mr S because he always walks you through step-by-step on how to do it and he gives you homework but he doesn't like overload you with homework and he doesn't make you rush, he'll let you take your time but you still, even though you take your time you still get all the work done without realising it."

Similarly, when the students entered Year 8, they still valued the teachers who had the ability to explain concepts clearly whilst acknowledging the diverse needs of the students, with one boy stating: "Sometimes when he explains something to us, it sometimes makes it easier for us. Maybe like drawing pictures or using one of the students to show how easy the work is."

Overall, the participants felt that a 'good' mathematics teacher was one with whom they could establish a positive pedagogical relationship with in order for them to feel as though they were able to learn. This quote sums up the feelings of the students:

> "well to me it depends on the teacher because if I don't like him or how he works I won't understand it. And if I do know, and I can have a bond with the teacher and just go up and ask him anything anytime and I understand it, I can get more work done and feel like I know something."

Over the course of the study the participants overwhelmingly found the pedagogical relationships with their mathematics teachers to be a powerful influence over their decision to engage in mathematics or not.

> "The good thing about maths is it really depends on the teacher. Like, I have a really good maths teacher and that's what makes maths fun because I can understand it."

This appeared to be more influential than the actual pedagogical repertoires (the day-to-day classroom practices) that teachers implement in their teaching of mathematics. The students' perceptions of their learning experiences will now be discussed.

What makes a 'good' mathematics lesson?

During Year 6, when asked to recall a 'fun' or 'good' mathematics lesson the majority of participants were able to quickly recall a lesson. Most of the lessons or activities cited as being the 'best' or most enjoyable were those that included physical activity, cooperative learning, active learning situations involving concrete materials, and games. Lessons that the students found particularly engaging were those in which they had a degree of choice within the tasks provided, giving them some control over their learning and a sense of ownership. Although all of the lessons described by the Year 6 students appeared to have been engaging, it is not possible to gauge the learning that occurred as a result of the activities.

When in Year 7, the participants found it much more difficult to recall a 'good' or 'fun' mathematics lesson, as the majority of the lessons were based on completion of tasks from a commercial mathematics website or the use of a standard text book on CD-rom. Students were also required to work individually, in contrast to their primary school experiences. Rather than recalling a memorable lesson, the students discussed the features they thought would make a 'perfect' mathematics lesson, which incorporated similar pedagogies to those included in their favourite primary lessons.

The inclusion of active learning was one aspect students desired, with one boy summing up the group's reflections:

> "I think a really good maths lesson for me would be being able to build something like with shapes for example as the lesson, but building something or being able to spend a lot of time and putting a lot of effort into something."

Variety also appeared to be an important desired element of mathematics lessons, with a Year 7 girl saying: "if the teacher gave us like a various amount of different work to do and you could choose which one you wanted to do." In addition to the pedagogies already mentioned in Year 6, the students articulated a desire to learn and understand the content presented in their mathematics lessons. A Year 7 girl said a good lesson would be: "if you know what you're doing and you understand it." Perhaps this was a sign of maturity or a sign that the students were beginning to become frustrated with existing routine pedagogies.

This frustration seemed to ease when the participants entered Year 8. The participants found it much easier to recall a 'good' lesson in this final phase of the study. Although it seems such lessons occurred infrequently, the engaging aspects of these lessons were the same as those that the students had recalled when they were in Year 6 and included games, doing mathematics outside the classroom and lessons that were a move away from a textbook approach.

It is worthy to note that the students did not include the use of technology in their perspectives of good mathematics lessons. This could be because of the way they had experienced their use, or simply because technology is so entrenched in their lives they do not consider it important in their learning. Further exploration of the use of technology and its ability to engage or disengage students is considered critical. However, it is beyond the scope of this paper.

Implications for the classroom

Many students in the final years of primary and early years of secondary education begin to develop a critical awareness of their own learning styles and pedagogical preferences. In this study, the participants' perspectives of 'good' teaching and 'good' mathematics lessons did not appear to change, although the teaching and pedagogies they experienced did change significantly.

Teachers remain a strong influence over student attitudes towards and engagement in mathematics. Most of the attributes of a 'good' teacher identified by students in this

study related to positive pedagogical relationships between the teacher and students. An understanding of how each child learns best, what each child knows and the ability to gauge how much scaffolding is required for students to learn is a foundation for positive student engagement in mathematics.

Students in this study showed preferences for tasks that required active learning, elements of choice, challenge and the option of self-directed activities. They also showed preferences for teachers who explained concepts or tasks well, and were aware of their students' needs and abilities. It is critical that teachers in the middle years address the specific needs of students during this transitional period. Identification of student preferences is one way that tasks and pedagogies can be tailored to the needs of students.

Many of the elements that the students claimed make a 'good' mathematics teacher are reflected in the AAMT's Standards for Excellence (2006). Elements of a 'good' mathematics lesson also reflect with the Productive Pedagogies (Lingard et al., 2001) and NSW Quality Teaching Framework (NSW Department of Education and Training, 2003), indicating a need for teachers to become familiar with and use these documents as a benchmark for quality teaching.

References

Askew, M., Brown, M., Rhodes, V., Johnson, D., & Wiliam, D. (1997). *Effective teachers of numeracy: Final report.* London: King's College.

Australian Association of Mathematics Teachers [AAMT]. (2006). Standards of Excellence in Teaching Mathematics in Australian Schools. Adelaide: Australian Association of Mathematics Teachers.

Barker, K., Dowson, M., & McInerney, D. M. (2005). *Effects between motivational goals, academic self-concept and academic achievement: What is the causal ordering?* Paper presented at the Australian Association of Educational Research (AARE).

Commonwealth of Australia. (2008). *National Numeracy Review Report.* Canberra, ACT.: Human Capital Working Group, Council of Australian Governments.

Gardner, H. (1993). *Frames of mind: The theory of multiple intelligences.* New York: Basic Books.

Haberman, M. (2005). *Star teachers: The ideology and best practice of effective teachers of diverse children and youth in poverty.* Houstan, TX: Haberman Educational Foundation.

Hayes, D., Mills, M., Christie, P., & Lingard, B. (2006). *Teachers and schooling making a difference.* Sydney: Allan & Unwin.

Lingard, R. L., Ladwig, J., Mills, M. D., Bahr, M. P., Chant, D. C., & Warry, M. (2001). *The Queensland School Reform Longitudinal Study (QSRLS).* Brisbane: State of Queensland (Department of Education).

Martin, A. J. (2008). *Motivation and Engagement Scale: High School (MES-HS) Test User Manual.* Sydney: Lifelong Achievement Group.

NSW Department of Education and Training. (2003). *Quality Teaching in NSW Public Schools.* Sydney: Professional Support and Curriculum Directorate.

State of Victoria Department of Education and Training. (2004). Middle years of schooling overview of Victorian Research 1998-2004. Retrieved July 7, 2005, from www.sofweb.vic.edu.au/mys/docs/research/

Sullivan, P., McDonough, A., & Harrison, R. T. (2004). *Students' perceptions of factors contributing to successful participation in mathematics.* Paper presented at the 28th Conference of the International Group for the Psychology of Mathematics Education.

LEARNING BY REFLECTING – A PEDAGOGY FOR ENGAGED MATHEMATICS LEARNING

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Reflection involves active, persistent and careful consideration of any form of knowledge. Students who are engaged in reflection learn more and find learning rewarding. Mathematics tasks are important vehicles for classroom instruction and through appropriate mathematical tasks teachers can engage their students in reflection. Some "What strategies" such as "What's wrong?, What's missing?, What's redundant and What's missing? have been found to evoke reflection amongst learners during mathematics lessons.

Pedagogy for engaged mathematics learning

Research has shown the importance of an engaging pedagogy (see Finn, 1993). Students who are engaged learn more, find learning rewarding and are more likely to continue to higher education. Engaging pedagogy contributes to social and cognitive development of students as well as achievement in academic pursuits they undertake.

Munns and Martin (2005) relate the psychological perspective of motivation and pedagogical perspective of engagement. They distinguish between the small e engagement and the big E Engagement. Attard (2009) describes the small e engagement as students busy working on procedures and participating in tasks while the big E Engagement as students perceiving doing mathematics as valuable.

A seminal study by Askew, Brown, Rhodes, Johnson and William (1997) found that highly engaging mathematics teachers shared some common characteristics. Amongst others, they used higher order thinking tasks to promote thought rather than practice. They also facilitated discussion of students' methods and reasoning, and used student responses to develop understanding and make connections within mathematics.

Pedagogies that have been found to be able to engage students are Learning by Inquiry, Learning by Doing, Learning by Interacting and Learning by Reflecting. It must be noted that these pedagogies are by no means mutually exclusive. This chapter focuses on one such pedagogy, i.e. Learning by Reflecting.

Learning by reflecting

Reflection involves "active, persistent and careful consideration" of any form of knowledge (Dewey, 1933). Reflection is useful in at least three ways. Firstly, it is important for students to be aware of their strengths and weaknesses as a learner. Secondly, reflection and metacognition are important when students solve problems. They need to evaluate if and how strategies that they have mastered can be used in the new problems. Thirdly, reflection is necessary for students to construct, refine and extend their knowledge.

In general education, there is the Kolb's (1984) experiential learning which includes reflective observation. In mathematics education, Polya's (1945) problem-solving model includes looking back which is essentially an exercise in reflection. Students who are reflective know if they need to draw a diagram to help them visualize a situation better. Students who are reflective understand that a problem is difficult and take greater care in communicating their solution methods. Teachers often insist that students write a statement after they have solved a problem because this is an act of reflection. A reflective student knows that there is something wrong if the statement is: Mr Lee is 165 m tall! Inevitably, students who are engaged in reflection learn more and find learning rewarding.

Some "what" strategies that facilitate reflection

Mathematics tasks are important vehicles for classroom instruction and through appropriate tasks teachers can engage their students in reflection. As part of a two year professional development project, Enhancing the Pedagogy of Mathematics Teachers (EPMT) in Singapore, teachers crafted mathematics tasks based on eight What Strategies (Kaur and Yeap, 2009a; 2009b). Some of the strategies have been found to have a strong reflection component (Yeap and Kaur, 2010). For example in "What's wrong?", students have to look at a solution and identify any mistakes in the solution. In "What's missing?", students have to identify missing information that are essential to solve a given problem. In "What's redundant?", students have to identify excess information that are not required to solve the given problem. In "What if?", students are asked to vary the variables in a given problem. Many students in the EPMT project were enthusiastic about such tasks.

Several of the teachers in the project wrote in their reflections that "many of them were very enthusiastic about it that they wanted to have this kind of tasks everyday".

What's wrong?

In, "What's wrong?" tasks students are provided with an opportunity to reflect and use their critical thinking skills. They are presented with a problem and its solution. However the solution contains an error, either conceptual or computational. The student's task is to recognize the error, correct it and then explain what was wrong, why it was wrong and what was done to correct the error (Krulik and Rudnick, 1999). Students may be asked to complete the task in small groups or individually. The teacher must ensure that students are engaged in class discussion after completing the task so that they get the opportunity to see ways of solving problems that differ from their own. Furthermore, these discussions often lead to deeper mathematical understanding (Krulik and Rudnick, 2001). Such tasks are not difficult for teachers to craft as they are constantly exposed to such errors students make in class and in their written assignments. Figure 1 shows an example of one such task that was used by a teacher in the EPMT project to engage her students in reflection and critical thinking (Yeap and Kaur, 2010, p.75).

What's wrong?

<u>Topic</u>: *Algebra*

1. Albert was asked to simplify $\frac{1}{c} - \frac{3}{2c}$

Albert's thinking:-

$$\frac{1}{c} - \frac{3}{2c} = \frac{2c-3}{2c^2}$$

There is something wrong with Albert's thinking.

a) Show how you would simplify the expression.

b) Explain the error in Albert's thinking.

2. Florence was asked to simplify $\frac{a+2b}{6} - \frac{a+b}{4}$

Florence's thinking: -

$$\frac{a+2b}{6} - \frac{a+b}{4}$$

$= 4(a + 2b) - 6(a + b)$
$= 4a + 8b - 6a - 6b$
$= -2a + 14b$

There is something wrong with Florence's thinking.
a) Show how you would simplify the expression.
b) Explain the error in Florence's thinking.

Figure 1

What's missing?

In "What's missing" tasks students are presented with problems that cannot be solved because an important piece of information has been omitted. Students must identify what is missing, supply appropriate data, and then solve the problem. Such tasks provide students with opportunities to engage in critical thinking, creative thinking and reflection. Whole class discussion must precede individuals working as pairs or small groups because the non-unique data suggested by the class for the missing information will lead to varied solutions. Students will actively engage in reflection when making sense of some of the solutions. Figure 2 shows an example of one such task that was used by a teacher in the project to engage his students in critical thinking and reflection (Yeap and Kaur, 2010, p.77)

What's missing?

Topic: *Trigonometry*
A ladder 6.5 m long leans against a vertical wall touching a window sill and making an angle with the ground. Find the height of the window sill above the ground. How far is the foot of the ladder from the foot of the wall?
The following prompts may be used by the teacher to guide the students if they are unable to work through the problem.
a) What are you asked to find?
b) What information do you need?
c) What information is missing?
d) Pick a reasonable value for the missing information.
e) Solve the problem.
f) Does your answer make sense?

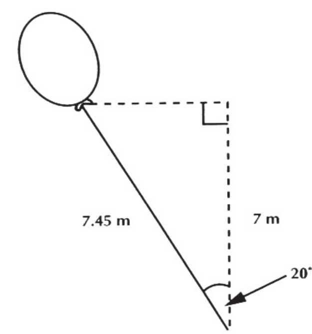
Figure 2

What's redundant?

In "What's redundant" tasks students are given extraneous information that is not needed for the solution of the problem. Students must identify the extraneous information and solve the problem with the appropriate givens. Such tasks engage students in critical thinking and reflection. Whole class discussion must precede students working individually or in groups on the task as it is important for students to reflect on the uniqueness or non-uniqueness of the redundant data depending on the nature of the task. Figure 3 shows an example of one such task that was used by a teacher in the project to engage his students in critical thinking and reflection (Yeap and Kaur, 2010, p.78).

What's redundant?

Topic: *Trigonometry (Pythagoras Theorem)*



A balloon is attached to a string which makes an angle of 200 with the vertical. The string is 7.45 m long and the balloon is 7 m above the ground. Find the horizontal distance of the balloon from the end of its string.

The following prompts may be used by the teacher to guide the students if they are unable to work through the problem.
a) What are you asked to find?
b) What information do you need?
c) What information is redundant?
d) Solve the problem.

Figure 3

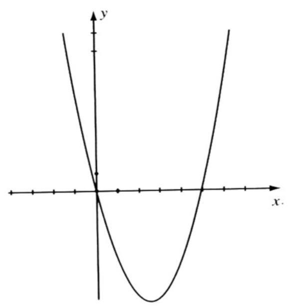
What if ?

In "What if ?" tasks students are systematically engaged in exploring what effect a change has on the solution process as well as the answer. In this way students are reinforcing their critical thinking and also reflecting as they analyze what is taking place (Krulik and Rudnick, 1999). Figure 4 shows an example of one such task that was used by a teacher in the project to engage her students in critical thinking and reflection (Yeap and Kaur, 2010, p. 76).

What if ?

Topic: *Quadratic functions*

The graph of the function $y = x\,(x - 5)$ is shown below.



How would the graph look like if the function is …

a) $y = x\,(x + 5)$

c) $y = x2 - 5$

e) $y = (x - 2)2 + 9$

g) $y = -\,(x - 2)2 + 9$

b) $y = x\,(5 - x)$

d) $y = (x - 3)(x + 3)$

f) $y = (x - 2)2 - 9$

h) $y = 3(x - 2)2 + 9$

Reflection

I. For graphs whose equation is written in the form of $y = (x - a)\,(x - b)$, the coordinates of the x-intercepts are ___________ and ____________.

II. For graphs whose equation is written in the form of $y = a(x - p)2 - q$, the maximum or minimum point has the coordinates _________________.

Figure 4

References

Askew, M., Brown, M., Rhodes, V., Johnson, D. & William, D. (1997). *Effective teachers of numeracy:* Final report. London: School of Education, King's College London.

Attard, C. (2009). Student perspective of mathematics teaching and learning in the upper primary classroom. In Cheah, U.H., Wahyudi, Devadason, R.P., Ng, K.T., Preechaporn, W., & Aligaen, J.C. (Eds.), *Proceedings of Third International Conference on Science and Mathematics Education,* pp. 246 – 254. Penang, Malaysia: SEAMEO-RECSAM.

Dewey, J. (1933). *How we think.* New York: D.C. Heath.

Finn, J.D. (1993). *School engagement and students at risk.* Washington, DC: National Centre for Education Statistics.

Kolb, D. (1984). *Experiential learning: Experience as the source of learning and development.* Englewood Cliffs, NJ: Prentice-Hall.

Kaur, B. & Yeap, B.H. (2009a). *Pathways to reasoning and communication in the primary school mathematics classroom.* Singapore: National Institute of Education.

Kaur, B. & Yeap, B.H. (2009b). *Pathways to reasoning and communication in the secondary school mathematics classroom.* Singapore: National Institute of Education.

Krulik, S. & Rudnick, J.A. (1999). Innovative tasks to improve critical and creative-thinking skills. In L. Stiff (Ed.), *Developing mathematical reasoning in grades K – 12,* pp. 138-145. VA: Reston, National Council of Teachers of Mathematics.

Krulik, S. & Rudnick, J.A. (2001). *Roads to reasoning – Developing thinking skills through problem solving [Grades 1 – 8].* Chicago, IL: Wright Group McGraw-Hill.

Munns, G. & Martin, A.J. (2005). *It's all about meE: A motivation and engagement framework.* Paper presented at the Australian Association for Academic research focus conference, Cairns, Australia.

Polya, G. (1945). *How to solve it.* Princeton, NJ: Princeton University Press.

Yeap, B.H. & Kaur, B. (2010). *Pedagogy for engaged mathematics learning.* Singapore: National Institute of Education.

DESIGNING MATHEMATICAL INVESTIGATIVE TASKS

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School teachers could be expected to design mathematical investigative tasks for their students. They may run out of ideas for designing such tasks. This chapter introduces different ways that teachers could consider in designing such tasks for their students by anchoring on the framework of mathematical problem solving.

Introduction

Before discussing mathematical investigation, it is crucial for readers to understand the position of mathematical investigation within a problem solving mathematics curriculum and a classification of mathematical tasks.

It is generally recognized that problem solving plays an important role in the learning of mathematics. In fact, there has been a world-wide push for problem solving to be the central focus of national mathematics curriculum. For example, the National Council of Teachers of Mathematics (NCTM) in their document on the principles and standards for school mathematics stated that "problem solving should be the central focus of the mathematics curriculum" (NCTM, 2000, p. 52). In Australia, the 1990 "National Statement on Mathematics for Australian Schools" stated that students should develop their capacity to use mathematics in solving problems individually and collaboratively (Australian Education Council, 1990).

What is a mathematics problem? According to the definition by Lester (1978), which is generally accepted by mathematics educators, a problem is a situation in which an individual or group is called upon to perform a task for which there is no readily accessible

algorithm which determines completely the method of solution. Lester (1980) added that this definition assumes a desire on the part of the individual or group to perform the task.

Based on the above definition of a mathematics problem, one could surmise that many mathematics textbook problems are not actually "problems" but are "mathematical tasks" that require students to apply suitable formulae and to practice the procedural skills that have been taught in the classrooms. These "problems" are more appropriately called "mathematical tasks", a term that we would use in this chapter.

A mathematical task can be classified as either routine or non-routine. A non-routine mathematical task can further be classified as either a "mathematical problem" or an "investigative task". Figure 1 presents a classification of mathematical tasks that will be adopted for discussion in this chapter.

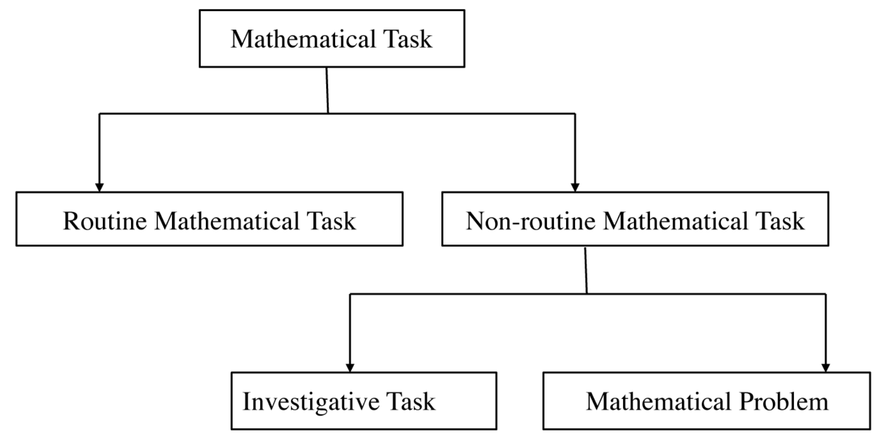


Figure 1. Classification of different types of mathematical tasks

The distinction between an investigative task and a mathematical problem has been made by researchers (for example, Frobisher, 1996). While a mathematical problem has a clearly defined goal (Henderson & Pingry, 1953), a mathematical investigation is open-ended and usually has ill-defined goal (Frobisher, 1996). Whether a task is routine or otherwise would very much depend on individual students (Henderson & Pingry, 1953). All this shows that such classification of mathematical tasks is not rigid. Nevertheless, we shall use the above classification for the convenience of our discussion.

Investigative Tasks in Mathematics Curriculum

Why do we belabor on designing investigative tasks when the worldwide focus of mathematics curriculum is on mathematical problem solving? A suitable answer to the above question could be seen from Bastow, Hughes, Kissane & Randall (1986): the importance of exploratory tasks lies in

> the subsequent solution of problems generated by the exploration highlights such mathematical processes as organizing, representing, specializing, pattern searching, generalizing, symbolizing, inferring, justifying and explaining. More generally, it provides opportunities for the development of independent mathematical thinking in learners, which is becoming increasingly important with the declining need for skill in mathematical techniques, now available on calculators and computers (p. 1)

The above quote reminds readers about the problem-solving heuristics and processes of mathematical problem solving popularized by Polya (1957). Investigative tasks, if appropriately designed, could actually facilitate students to build up their problem solving competency by instilling in them the application of the various stages of problem solving, especially the application of the problem solving heuristics and developing their mathematical thinking skills.

Designing Investigative Tasks

Teachers occasionally run out of ideas in designing new investigative tasks for their students. The author proposes four approaches of designing new investigative tasks, the ideas and examples of which will be presented in this chapter.

Approach One: Based on Mathematics Education Literature

Ideas of many interesting investigative tasks can be obtained from literature on mathematics education. For example, literature on children's learning of mathematics generally shows there is a lack of understanding about the attribute that is being measured (for example, Outhred & Mitchelmore, 2000; Koay, 2007). Anecdotal evidence in classrooms shows that students apply wrong formulae for area and perimeter of a rectangle. The author's personal experience shows that secondary four students (ages 16 to 17) have the misconception that area of rectangles is proportional to their perimeter. Had

students been given sufficient exposure to investigative work on areas and perimeters, such misconceptions could have been averted or reduced. Consider the following suggested investigative task, which is suitable for lower secondary students (ages 12 to 14):

Task A: A rectangle has an area of 36cm2. Investigate its perimeter.

Commentary on Task A: Students would identify that there is no unique rectangle with an area of 36cm2 (Understand the Problem – Polya's Stage I). They would then attempt to draw different rectangles with area 36cm2 (substituting different values – a problem solving heuristic). By calculation, it would be observed that all these rectangles have different perimeters, even though they have equal areas (discover that areas and perimeters do not necessarily correlate). Further pattern generalization reveals that the perimeter is minimal when the rectangle becomes a square (where both the length and breadth are equal), and there is no limit as to the maximum perimeter to be achieved with an area of 36cm2. This could even lead students to question further the relationship between a rectangle and a square, a higher level of Van Hiele's stages of students' learning of geometry (Leong & Lim-Teo, 2007). Not only does this investigative task lead one to go through various stages of mathematical problem solving, it also leads one to a higher level of the learning of geometry.

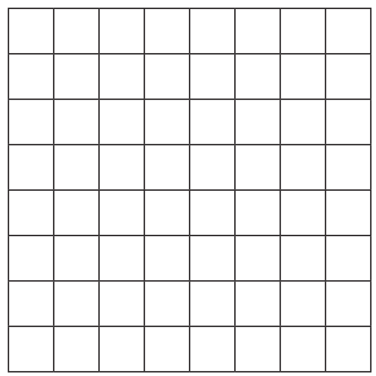
Approach Two: Based on Higher Order Thinking Questions

Many higher order thinking problems can be converted into investigative tasks. For example, consider the problem:

Question B. Find the number of squares in an 8x8 square board.

The solution of question B can be a good illustration of the application of Polya's stages of problem solving (Toh, Quek & Tay, 2007). This question can easily be converted into an investigative task, which widens the scope of students' exploration of the given geometrical figure.

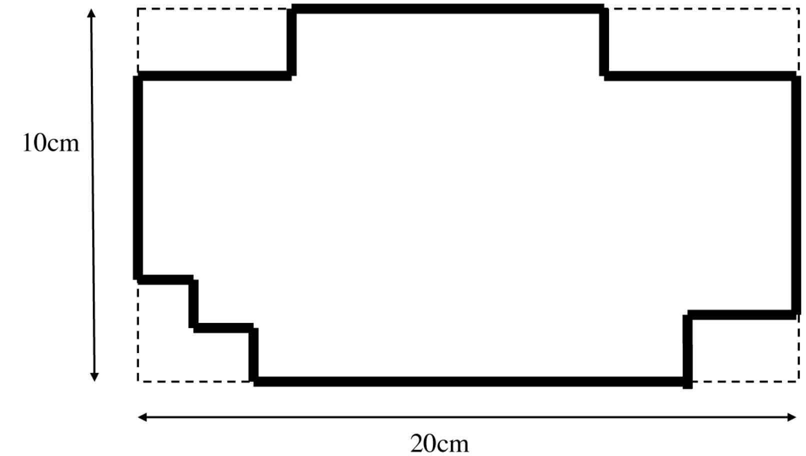
Task B: The following diagram shows an 8x8 square board. Investigate the geometrical shapes.



Commentary on Task B: This task allows students to have deeper investigation compared to the original question B. It is crucial for teachers to know at least several different approaches to investigate this problem. For one, students could use this problem as one involving the counting of squares (Toh, Quek & Tay, 2007); another approach to explore this problem would be the identification of the number of rectangles (which obviously would be more than the number of squares). In fact, the diagram of an 8x8 square board can be the context of many problems on combinatorics.

This task demonstrates the richness of the context of the original question B when it is converted into an investigative task B. It is important that teachers are able to facilitate students to progress along the various approaches of investigating the problem by providing various platforms of scaffolding.

Another illustration of converting a higher order thinking question into an investigative task will be presented below. The following is a popular mathematics competition question in the Singapore Mathematical Olympiad.



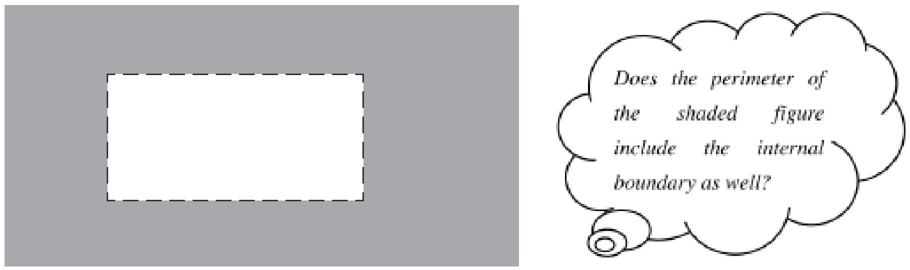
Question C. Part of a rectangle of length 20cm by 10cm is removed by cutting off lengths which are parallel to the two sides of the rectangle as shown in the following diagram. Find the perimeter of the remaining shape.

Question C leads students to identify the conservation of perimeter from the original rectangle, even though some parts of the rectangle have been removed. Removing parts of the rectangle does not necessarily decrease its perimeter – it depends on how the rectangle is cut. This question could be turned into one engaging students to explore the relationship between area and perimeter of plane figures; see Task C.

Task C. You are given a rectangle 20cm by 10cm, and allowed to cut away parts of the rectangle. Investigate the perimeter of the remaining figure.

Commentary on Task C: By cutting the corners of the rectangle with the removed sides parallel to the sides of the rectangle, students would observe that the perimeter of the remaining figure is conserved. By cutting the corners with at least one side not parallel to the two sides of the triangle, the perimeter of the remaining figure is decreased. This is a corollary of the fact that the side of any side of a triangle is always less than the sum of the two other sides. Further, teachers could challenge their students to cut the rectangle such that the remaining figure has a bigger perimeter than the original rectangle!

Students could even be led to think more deeply on the concept of perimeter. In the process of allowing students to cut into any shapes they like, they may end up with a concave figure. For example, when the author worked with a group of practicing teachers in a professional development course, one participant ended up struggling on how the perimeter of a concave figure (Figure 2) is defined – should the perimeter of Figure 2 be the sum of all the boundaries?



As task C could lead students to think relationally on the concept of perimeters, teachers should provide different levels of scaffolding to lead their students to different levels of understanding according to their ability.

Approach Three: Based on Students' Mistakes

Anecdotal evidence from classrooms shows that students make many mistakes in school mathematics due to their acquisition of procedural knowledge which is not backed by conceptual understanding (Yeap, 2009, p. 32). This is especially true in students' learning of school algebra. A very common mistake among students is the algebraic expansion:

$$(a+b)^2 = a^2 + b^2$$

Teachers would frequently demonstrate to students that in fact the above expansion is incorrect (see, for example, Yeap, 2009, Worksheet 1 & 4).

After leading students to appreciate the mistake of the above expansion, teachers could create the opportunity for their students to investigate expansion in general:

Task D: Consider $(a+b)^n$ and $a^n + b^n$. Investigate.

Commentary on Task D: This task has different approaches of investigation. First, students could see that the two algebraic expressions are generally different, except for n = 1. When n = 2, the two expressions are not equal. This question could serve as a

consolidation of the basic algebraic expansion of and further arouse students' curiosity on the expansion of for larger integral values of n (which is the well-known binomial expansion for positive integral powers). Recently, researchers have started to research on the importance of curiosity to the learning of mathematics (Toh, 2009). Teachers could further challenge their students: What can one conclude (about a and / or b) if one knows that the two algebraic expressions are equal for the case n = 2? What about n = 3? Can you justify your conjecture?

Task D could begin as consolidating students' basic algebraic concepts, leads on to arouse their curiosity to learn more about algebraic expansion and lead them to higher order thinking questions.

There are many other mistakes that students could make in learning school mathematics. As illustrated above, these mistakes could be transformed into opportunities for students to perform mathematical investigation.

Approach Four: Based on Routine Mathematical Content

There are many mathematical concepts that students learn procedurally without much conceptual understanding or any appreciation of these concepts. The teachers can convert some of these concepts into investigative activities for their students. One example of such a concept is logarithm (Toh, 2009).

Students invariably learn how to compute logarithms using definition, calculators and solve logarithmic equations required in the mathematics curriculum. Seldom do they realize that the common logarithm of an integer provides information about the number of digits needed to write the number in its decimal representation (Toh, 2009). An investigative activity could be designed for students to appreciate such mathematical concepts.

Task E: Investigate $\log_{10}x$ for different integer values of x.

Commentary on Task E: Teachers can first get students to carry out Task E with basic calculating device, and then without such a device. Students could be guided to think over the question: "What does the value of $\log_{10}x$ indicate about the value x?" This develops their skill in pattern generalization – an important problem solving heuristics crucial in problem solving. For the higher ability students, teachers could even change the investigation to log2x or logarithm with any other positive integral bases (except 1).

One may argue that Task E is NOT a mathematical investigation since it is not exactly open-ended and the goal is rather well-defined (Frobisher, 1996; Henderson & Pingry, 1953); it however contains the elements of mathematical investigations when the end goal is not made explicit, hence enabling students to explore. It is crucial to engage students in non-routine exploratory tasks which could lead them to appreciate mathematics, stretch their higher order thinking skills, and, most importantly, this element of investigation could further develop students' mathematical problem solving skills,

Conclusion

In this chapter, the author has outlined four main approaches that teachers can take in designing mathematical investigation tasks for their students.

The author conducted a professional development workshop on mathematical investigative tasks for thirty practicing teachers (Toh, 2007). The participants were amazed at the richness of such investigative tasks. However, it should also be noted that teachers must be sufficiently competent in both mathematical content knowledge and pedagogical content knowledge in order to maximize students' benefits through such potentially rich investigations.

References

Australian Education Council. (1990). *A national statement on mathematics for Australian schools*. Australia: Curriculum Corporation for Australian Education Council.

Bastow, B., Hughes, J., Kissane, B., & Randall, R. (1986). *Another 20 mathematical investigations.* Perth: The Mathematical Association of Western Australia.

Frobischer, L. (1994). Problems, investigations and an investigative approach. In A. Orton, & G. Wain (Eds.), *Issues in teaching mathematics* (pp. 150 – 173). London: Cassell.

Henderson, K. B., & Pingry, R.E. (1953). Problem solving in mathematics. In H.F. Fehr (Ed.), *The learning of mathematics: Its theory and practice* (pp. 228 – 270). Washington DC: NCTM.

Koay, P.L. (2009). Teaching of measurement. In P.Y. Lee & N.H. Lee (Eds.), *Teaching primary school mathematics: A resource book* (pp. 199 – 225). Singapore: McGraw-Hill.

Lester, F.K. (1978). Mathematics problem solving in the elementary school: Some educational and psychological considerations. In L.L. Hatfield & D.A. Bradbard (Eds.), *Mathematical problem solving: Papers from a research workshop* (pp. 53 – 88). Columbus, Ohio: ERIC/SMEAC.

Lester, F.K. (1980). Research on mathematical problem. In R.J. Shumway (Ed.), *Research in Mathematics Education* (pp. 286 – 323). Reston, Virginia: NCTM.

Leong, Y.H., & Lim-Teo, S.K. (2009). Teaching of Geometry. In P.Y. Lee & N.H. Lee (Eds.), *Teaching secondary school mathematics: A resource book* (pp. 25 – 50). Singapore: McGraw-Hill.

NCTM (National Council of Teachers of Mathematics). (2000). *Principles and standards for school mathematics*. Reston, VA: NCTM.

Outhred, L.N., & Mitchelmore, M.C. (2000). Young children's intuitive understanding of rectangular area measurement. *Journal for Research in Mathematics Education*, 31(2), 144 – 167.

Polya, G. (1945). *How to solve it. Princeton*: Princeton University Press.

Toh, T.L. (2007). An in-service teachers' workshop on mathematical problem solving through activity-based learning. *Journal of Science and Mathematics Education in South East Asia*, 30(2), 73 – 89.

Toh, T.L. (2009). Arousing students' curiosity and mathematical problem solving. In B. Kaur, B.H. Yeap, & M. Kapur (Eds.), *Mathematical problem solving: Association of Mathematics Educators Yearbook 2009* (pp. 241 – 252). Singapore: World Scientific.

Toh, T.L., Quek, K.S., & Tay, E.G. (2007). *Problem solving in the mathematics classroom (Junior College)*. Singapore: Association of Mathematics Educators.

Yeap, B.H. (2009). Teaching of Algebra. In Lee P.Y., Lee N.H. (Eds), *Teaching Secondary School Mathematics: A Resource Book* (pp. 25 – 50). Singapore: Mc-Graw Hill.

DIFFERENTIATING MATHS TEACHING AT SECONDARY LEVEL

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This article makes a case for the use of structured differentiation in classrooms, to accommodate the varied educational needs of the students. It is argued that group work is an essential ingredient, both mixed groups for investigations and like-need groups for targeted teaching.

Some history

Half a century or more ago those who could not cope with the rigors of secondary level maths left school at 14 (for example, my mother), or took an apprenticeship (for example, my uncle). A few went to one of the small number of technical colleges (for example, my father, my brother), and most of the remainder dropped out before 'matriculation' (for example, my wife). A very few went to university (for example, myself).

This was the way we differentiated in the past. There was a 'standard' for each year level, and those who did not meet it simply dropped out.

In the generation that followed there were attempts to escape the domination of standards by several new styles of organising resources. The best-known were the 'individualised' programs. These were either booklets or cards that were little more than a textbook chopped into small pieces. Students worked through these 'at their own pace', which tended to be much slower than desirable. The better readers coped well, because they could interpret the booklet or card, but the less able readers were constantly either calling for teacher support or just wasting time. This model soon disappeared because it did not work.

Part of the reason for the present interest in differentiation is that the attitude that puts 'arbitrary standards' ahead of the needs of individual students is *no longer acceptable*. It is now regarded as a responsibility of all schools to make sure that *all* students continue to

learn from wherever they happen to be, mathematically. We no longer have the option of letting them drop out.

Text-book style teaching

The style of teaching that accompanied the historical model described above was dominated by textbooks and teacher-dominated instruction. The topic was taught from the front of the room, and the students coped as well as they could. Then they practised the skills that were demonstrated (frequently without theory or understanding) using a textbook. The book set the standard and supported the teacher, who often depended on the explanation and worked example that came before the exercises. Work not completed in class was set for homework, because the next day the class moved on to the next section. Those who could not keep up simply failed and dropped out.

Textbooks from that era assumed that they had the responsibility to interpret the 'syllabus'. They did it efficiently, but took the approach that 'one-size-fits-all'. This is not a criticism of the authors from the days when textbooks set the standard, because in those days the system depended on good books to keep it going. Those of us who survived the system, and even became maths teachers, were clearly part of a success story. But there were many who failed, learned to dislike maths, and dropped out.

Since those days the basic assumption about the responsibility of schools and teachers have changed. Students are not dropping out, and it is government policy that they stay on as long as they can. It is now expected that *all* students will continue to learn productively, and textbooks that support the old attitude are out of step.

Yet in visiting schools I often hear the complaint that at least half of the students cannot use the textbooks effectively. Either students cannot (or will not) read it, can read it but cannot understand the explanations or cannot do the level of maths required for success. For many students and their teachers, the textbook has become a barrier to success, not the means to achieve it.

We still have many students at a level that would have led to them leaving school (or at least dropping maths) 50 years ago. So there is a need for a different style of teaching to cope with the increasingly wide range of levels in the classroom. In the rest of this article I will explore some of the aspects of a different style of teaching that does work, because it differentiates. The more mathematically advanced can be encouraged to fly, but not at the cost of those who are still walking. The less mathematically advanced can walk and learn to run, but not at the cost of preventing those who could fly from doing so.

Critical ways students can differ

Mathematical level

From entering school students differ considerably in background and ability to learn mathematics. Throughout primary school it is quite unrealistic for teachers to expect the same level of achievement from all. The reports from teachers at the end of primary school indicate a range of at least six or seven chronological years (three or four VELS levels).

Yet there are still many secondary schools where all these students are sold the same textbook, and marched through the same material at the same speed, as if these differences in mathematical achievement did not exist. Because the 'traditional teaching style' is dominated by the one-standard textbook, and because some books of that style are still being published, teachers are caught in a bind. The school administration wants them to be more flexible, but they are tied to an expensive book paid for by parents or guardians. These people do not understand that the assumption about 'standards' has changed, as their only memories of maths are related to their own success or failure in the old teaching style. There is an alternative – keep reading.

Success and self-esteem

Students differ greatly in their level of success in the 'standard-dominated' style of teaching. Those who are failing lose self-esteem and can often become the disrupters that make teaching the others far more difficult. Telling the 'failures' to sit quietly and do something else might be a way to allow the teacher to survive, but does not meet the obligation we have to ensure that all students continue to learn.

High self-esteem is most closely related to success and the positive feedback that comes from success. Even those we deem as failures want to succeed, and it is that success that will turn their self-esteem around.

Understanding vs rote learning

There appear to be two 'extreme' kinds of 'successful' learners. The most common are those who have learned to survive in maths classes by doing what they are told. Their 'success' means that they can do what they are shown, and get good results on tests, provided these are given immediately after learning that section of the work. But the learning is frequently short-term, forgotten shortly after. Textbooks seem to be written on the assumption that most students are in this category. There is a huge amount of overlap between content at different year levels, and this contributes to the view that the curriculum is over-crowded. Having to reteach previously forgotten work takes much of our time.

At the other extreme are those who seek to understand the maths and relate new ideas to the other ideas already grasped. These are the mathematical thinkers, who are learning to become mathematicians. Their emphasis is on 'working mathematically' and because of their active mental engagement they really understand and enjoy mathematics. They can also apply what they have internalised to new situations. They enjoy mathematics competitions, puzzles and logic. Most teachers are in this category. Problem solving activities and investigations are the ways to encourage everyone to join this relatively rare type of student.

Ways of differentiating teaching

Teach for understanding

The first step is to set our goals high. We want all students to understand, to relate new ideas to old ones, and to engage with the ideas, not the skills. This is not to denigrate skill learning, but to see it as a consequence of understanding, not as a replacement for it.

Teaching for understanding requires focusing on the concepts and using 'hands-on' materials, where these are available, to give many students a chance to think out the concepts for themselves. 'Hands-on' materials are best used in a group situation where students help each other, and teacher support is given to the group, not to individuals. To put it another way, for the majority of students understanding is best achieved by getting them engaged in the active process of thinking – working mathematically. This develops confidence and problem solving, self-esteem and language.

The argument is often presented that there is not enough time to use materials. In fact materials lead to permanent understanding, not short-term routines. The time needed to practise related skills is reduced, as is the need to reteach from year to year. This means that no extra time is needed – provided that the understandings are developed well at the level and at the time the student is ready to learn them. Such understanding also leads to increased confidence, self-esteem and increased motivation.

Use a variety of learning styles and groupings

Given that students differ in their preferred learning styles, the best a teacher can offer is variety. Here is a short list: whole-class instructions, pep-talks, games and activities such as investigations; small-group games and activities such as problem solving tasks; computer activities – either alone or in pairs; individual and group library or internet projects; excursions and guest speakers; DVDs and plenty of chance for informal talk. Use the space outside the classroom. Make lessons memorable.

Accommodate a wide range of mathematical levels

Since most classrooms have students covering a range of at least six or seven chronological years (three VELS levels) it is pointless to treat students as if they were at the same level. There are two main ways this differentiation can be achieved: open-ended tasks, and targeted teaching.

Open-ended tasks are designed to have multiple starting points and multiple end points. There is often more than one answer, but certainly more than one way to get an answer. Sources of these include RIME, Maths300, and problem-solving tasks from the Mathematics Task Centre. However many closed questions can be turned into open ones merely by providing the answer and asking for the question.

Targeted teaching means short-term teacher instruction with a small group to meet an immediate common need of that group. Groups at several levels may be formed in the classroom to allow this to happen. Of course, while a teacher is working with one group the rest of the class has to be supervised and usefully engaged in their own tasks – at their own level.

Highlight applications and relevance

Many disenchanted students have been known to respond positively to realistic real-life applications of mathematics. One way to organise content for such students is the thematic unit. The real life application context provides a vehicle for mathematical learning and this is seen as relevant. Such a unit for country students might be the mathematics of running a farm, where city students might study the mathematics of public transport. For all students units on the mathematics of pop-music, football (any code), gambling, money, moving out of home, cars, food (and many others) can get them wanting to come to maths class. However teachers should be warned that many applications require very specific contextual knowledge for the mathematical applications to be understood, and this may be enough to discourage students from the engagement. It is best to offer students a wide choice to accommodate the diversity of interests.

At a more sophisticated but more useful level students can be taught aspects of mathematical modelling. This means they learn to look for the type of mathematics that is involved in a wide range of different contexts, paying less attention to the contexts but more to the mathematical structure of the problems involved. This means that we enhance the ability to relate mathematical ideas and skills to problems likely to be encountered in the future. The more limited alternative is a set of applications to a set of real-life contexts which are likely to be of interest to current teenagers.

Organising classrooms for differentiation

Assessment tools

There are many tools now provided for teachers to get to know the mathematical knowledge of their students. These are provided free on relevant websits. Many are for use one-on-one, and some give specific advice on how to respond to the student's responses.

- On-Demand Testing, previous NAPLAN tests, previous AIM tests (VCAA)
- Fractions and Decimals Online Interview (DEECD)
- Scaffolding Numeracy in the Middle Years, emphasising multiplicative thinking (DEECD)
- Assessment for Common Misunderstandings (DEECD)

In addition a teacher will have access to the previous teacher's reports, and to their own observations, samples of student work and test results. Additional information can be obtained from student self-assessment, such as interviews, surveys, log-books and journals.

With adequate knowledge of the mathematical background of your students it is relatively easy to place them into groups with similar needs.

Use group work, flexibly

There are many virtues of group work. Groups work best with three or four but sometimes two-person games are useful; larger groups tend to break up into smaller groups. For investigations and open-ended tasks, groups should be deliberately mixed in mathematical level, including at least one good reader. For targeted teaching, groups should be organised according to mathematical level. My personal experience suggests that teenagers often feel the need to work with a particular friend, and that it is not always wise to oppose this. Cooperation is more important than uniform groups.

The MAV differentiated unit plans (2011 edition)

The MAV is promoting a package of planning resources, supplied to members both on our website and on a CD called 'Teach Maths for Understanding' - sent to members in the first annual mail-out. These resources can make differentiated teaching a feasible option from the beginning of primary school to Year 10. This section will describe these, but teachers must be aware that having the resources is only a part of the picture of differentiation described above. You need to find a way to adapt them to your own situation.

Mix of open-ended and targeted teaching

The MAV Differentiated Unit Plans provide plans for a balanced curriculum at all levels. All content dimensions are 'covered' but always at the level of the student. Less time is required on each topic than traditionally because the students are learning only at one level, not attempting the three or more levels in a full classroom.

The teaching styles are a mix of investigations (largely Maths300 or RIME), and various forms of instruction and practice. While a teacher is working with a group at one table (i.e. at one level) the others have specific worksheets or other resources to use – always at their own level.

Grouping of students by need

For the investigations mixed-level groups are recommended, but for the targeted teaching the instruction (often using hands-on materials) will be to small groups at the 'same' level. Teachers can use many of the forms of assessment suggested above to guide the formation of groups with similar needs.

Variety of learning styles

The plans have a particular structure. Each unit starts with assessment and an open-ended investigation, to allow the teacher to get the feel of where each student might be placed for the 'grouping by need' days that follow. Assuming three levels in the class, there is a three-day rotation. Teaching will be to the lowest level first, then the middle group and then the highest level. While this happens others will be working on worksheets and other tasks (such as problem solving) pitched at their particular level.

Following the three-day rotation the class might go to a computer lab, or use laptop trolleys or a pod. The plans suggest spreadsheets, Learning Objects or applets (from the internet) for this purpose.

Hyperlinks to a wide range of resources

If the plans were only ideas about what should be taught, they would be very difficult to achieve, such the resources would be missing. However MAV has hyperlinked the various sections of each plan to specific resources to make it achievable. Many of these are external to MAV and link via the internet. They include all the many free resources from DEECD, NAPLAN, Maths300, Mathematics Task Centre and NCTM. The others are to resources sold by MAV, and currently available on CD. There are also books referenced, unavailable for hyperlinking.

Wherever the plans suggest teaching to a small group, there is a hyperlink to teaching ideas (usually 'hands-on') from files on the CD called 'Teach for Understanding'. Even if you feel unable to use grouping by need within your classroom, these teaching ideas may well offer either new thoughts or a reminder of ideas you have previously used.

Many teachers will have knowledge of many good resources that are not sold through MAV and will wish to add to the resources listed; this is to be encouraged.

Where does the textbook fit?

There will be some 'academically-inclined' students who learn very successfully from such a resource aimed at their year level or even the year above. For the others, many of whom cannot use much of their year-level book, make good use of the parts that are useful. Cut out pages from the Year 7 book for students in Years 8, 9 or 10, and remove evidence of its level if that might restrict their willingness to use it.

When the Australian Curriculum is finalised, all the plans on this CD will be adapted to the new structure. In this way teachers and schools will be able to differentiate their maths teaching well into the future.

GEOCACHING: A WORLDWIDE TREASURE HUNT ENHANCING THE MATHEMATICS CLASSROOM

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Geocaching is a global treasure hunt that invites people of all ages to discover actively the beauty of their environment through the assistance of a Global Positioning System (GPS), mathematical know-how, and a bit of foraging. If you are seeking a new way to engage your students in a motivating and exciting real-life task, then geocaching might be the answer. The purpose of this article is to describe the experience of our school-based geocaching project undertaken with children in Prep (ages 5-6) and the senior primary Grades (ages 10-12). We will share the potential for mathematical learning and engagement. It is argued that geocaching provides the opportunity for rich engagement with key mathematical concepts that goes beyond what can be achieved during a typical lesson.

What is Geocaching?

Geocaching is a high-tech worldwide treasure hunt played throughout the world by adventure seekers of all ages with a strong sense of community and support for the environment (www.geocaching.com accessed July 21 2010). Geocache comes from the term Geo = *earth* and Cache = *hidden* supply or treasure (Christie, 2007). The 'geocacher' is equipped with a GPS device (see Figure 1), locates hidden containers called 'geocaches' in the environment and shares his/her experience online with the geocaching community. Geocaching was started in 2000 by a technology enthusiast in Oregon, U.S.A. wanting to test out the accuracy of his GPS. The bug for hiding and finding treasures soon took off with fellow GPS enthusiasts in hot pursuit. Today the leading geocaching website geocaching.com boasts over a million active geocaches around the world (http://www.geocaching.com accessed 21st July 2010). A geocache is typically a small watertight container that holds a log book, pencil and little 'treasures'. The treasures inside the geocache are usually inexpensive toys and trinkets. The geocache location coordinates and clues are posted on the internet for interested seekers. The understood protocol is that the finder enters their name in the log book, takes one item and replaces it with something of equal or greater value, and re-hides the geocache for the next geocacher. The seeker logs his/her find online for others to view.

Figure 1 - Children geocaching

Why Geocaching?

Many mathematics educators support the view that we should make mathematics real for students (Sparrow, 2008) and this is reflected in national and local curriculum documents. As Gerofsky (1996) noted, some mathematics educators have taken this challenge to mean creating higher quality and more varied word problems, the view being that making the connections to real-life situations is undertaken through students engaging in word problems. Yet there is evidence that suggests that some children who successfully perform mathematical problems in the 'real-world' are unable to solve word problems in a classroom context (Nunes, Schliemann, & Carrere, 1993). As Ainley (2004) suggested 'What is needed is more careful attention to designing and implementing conditions that maximize the opportunity for lively, challenging learning experiences.' The Geocaching Program goes beyond the creation of word problems to provide a real-life connection but, rather, attempts to maximize the opportunity for lively learning experiences through linking classroom based learning to a community-based activity that goes beyond the school walls.

Geocaching offers 'real and relevant mathematics' (Sparrow, 2008, p.4) that connects a need to develop spatial skills to successfully engage in treasure hunting. The geocacher requires an understanding and awareness of the functions of a GPS, compass points, longitude and latitude, distance, reading and following maps, trip planning, drawing and using scale, interpreting data, and along with literacy and ICT skills, undertakes physical exercise. Geocaching provides a link to fellow members of the community who are stimulated by treasure hunting and enjoy outdoor pursuits. Geocachers typically hide their geocache in a location that holds importance for them, e.g. along a favourite walking trail. The geocache seeker is provided with an insight into an area that may only be known to locals. As the seeker, you are able to see the world through another person's eyes. For the hider there is an inherently satisfying feeling knowing you have given others the opportunity to share a location that is special to you.

Geocaching in our school community

Wooranna Park Primary School has placed a focus on the development of mathematical tasks that support a social constructivist perspective which originates from Piaget's (1937) theory of the child's cognitive development. An understanding of the world is constructed rather than passively received. Our aim was to find a vehicle for learning that would allow for the students to actively construct their mathematical knowledge through making mathematics relevant, engaging and challenging. The project groups included students from both the Prep class and the 5/6 unit. It was decided that by involving the youngest and the eldest children of our school community we would be better able to understand the potential for geocaching to enhance mathematical understanding across the primary school setting.

At the planning stage teachers and mentor (Bragg) discussed the mathematical skills the students would need, and the understandings they required to find independently a geocache. The major curriculum focus became the development of spatial awareness, language and mathematical skills related to measurement, space, location and geography. Our general focus points were derived from the Australian Curriculum Draft (Australian Curriculum Assessment and Reporting Authority, 2010):

- *To assist students in becoming confident, creative users and communicators of mathematics.*

- *To provide children with the opportunity to pose basic mathematical questions about their world and to strengthen their reasoning to solve personally meaningful problems.*
- *To provide experiences that will ensure the development of an increasingly sophisticated understanding of mathematical concepts related to measurement, space, location and geometry.*

A series of lessons was developed focusing on the content descriptors for Measurement and Geometry content strand of the Australian Curriculum Draft (Australian Curriculum Assessment and Reporting Authority, 2010). The main curriculum focus was to improve students' directional language when explaining location and the ability to read and produce accurate maps. The affective focus in the senior classes was engagement in mathematics and to promote mathematics in a real life context that was transferable from the classroom to community based situations. The Grade 5 and 6 students were chosen randomly across three class groups of 28. These students possessed a range of capabilities and varying interest levels from the highly motivated to the mathematical disengaged.

The focus in the Prep area was to provide experiences that would further develop students' spatial concepts. We encouraged the children to draw from the world around them and to build on their experiences by focusing on spatial aspects of their immediate environment. The results of the Early Year's Interview (Detour) (The Department of Education and Early Childhood Development, 2010), was used to help select an initial 18 children for the project. All children nominated to join the group were able to manipulate the 'teddies' in the interview and use the appropriate language when articulating the position of the teddy, for example the blue teddy is *next to, behind, in front of* the yellow teddy. Their ability to order numbers and count collections displayed a solid foundation in number sense.

The Geocaching Program was launched as a secretive mission that was only to be discussed with family and friends. One of the key aspects of geocaching is that the location of geocaches should remain a secret and only revealed to fellow geocachers. The students were introduced to geocaching through a variety of mediums. The students viewed internet videos which were designed to explain the geocaching experience. Teachers used personal geocaching adventures to stimulate interest and curiosity by immersing students with stories and photos. The students were also given time to explore the geocaching website focusing primarily on frequently asked questions and how to get started. The students explored handheld GPS devices within the school grounds adding to the engagement and excitement about the project. Time was provided for students to discuss the information with their peers and brainstorm questions about geocaching. For example, some of the students enquired, 'Is it hard to do geocaching?' and 'What is going to be in a geocache?' The teachers found that geocaching, as a vehicle for learning, enabled students to engage immediately and become interested in the prospect of using mathematics to find hidden treasure.

All lessons were designed to build on prior knowledge through hands on, exploratory tasks with a clear mathematical focus. Students were challenged to learn new concepts by working collaboratively with all participants both inside and outside the classroom environment. The Grade 5 and 6 students were responsible for mapping and hiding six geocaches around the school grounds. They created accurate scale representations of the school buildings and wrote directions using mathematical language to support the Prep children locating the 'treasures' (see Figure 2.).

The older students integrated learnt skills in compass directions and map reading to aid the younger students in successfully completing the task. The Preps built on their spatial awareness and the effective use of directional language through conversations with the older students, which included statements such as 'we need to turn left' and 'go forward 10 steps.'

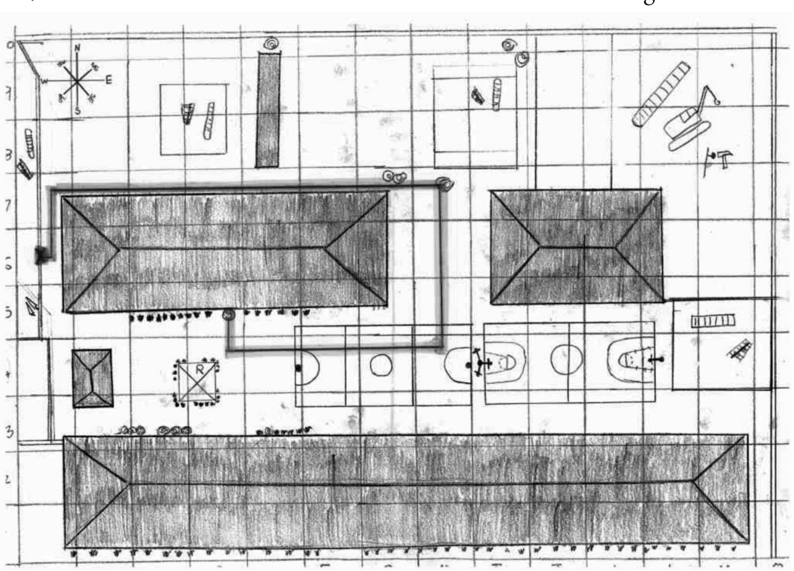

Figure 2 - Map to the in-school geocache

Early discussions with the Preps and initial drawings of maps highlighted a broad range of skills and conceptual development. The need to enrich spatial awareness and to further explore the understanding of directional language and mapping skills was apparent; an understanding of location can be a daunting challenge even for the adult mind. The use of still and flip video cameras to capture learning spaces and the classroom environment assisted in creating a larger, more concise map of the area and helped to further develop their spatial visualisation and spatial reasoning ability. Though this was a demanding task, the children remained focused, worked together, shared understandings and persisted with the task.

The children's greatest challenge was to employ effectively their directional language when required to physically move and consider their position within the classroom space. *Left* and *right*, *forward* and *backward* were terms the children needed to rediscover. The Bee Bot© was employed to assist the children's understanding of direction. The children placed arrows on the floor to plan a route. They walked the arrow route, verbally described their movements and programmed the Bee Bot© to recreate their steps (see Figure 3).

The culminating experience for students was an expedition to a local geocache hidden in the Dandenong area called 'Bird N Worm'. The logistics of the excursion were the biggest challenge with permission forms, car seats and finding eligible drivers taking most of the organisational time. We decided on manageable group size of 10 students; five seniors and five juniors. We supplied one GPS per pair of students and incorporated a parent's iPhone with a downloaded geocaching application available to support the search. The level of excitement was evident from the moment the students arrived at school that day. The teachers were impressed by the confident manner in which all the students handled the GPS and interpreted effectively the information it provided. The students referred regularly to the GPS as they walked and were able to relay confidently co-ordinate information, directed each other using compass points, and offered regular updates on distances to be covered to the geocache.

Figure 3 - Bee Bot in action

Assessing mathematical learning of geocaching

The students' mathematical learning in the Geocaching Program was assessed in a variety of ways, such as, student interviews, teacher observations, anecdotal notes, student work samples and reflective journals. Initially all students were requested to draw a map of their path to school and use directional language to describe the route. The mapping task was used as a formative pre-assessment task to gauge the students' knowledge, and as a tool for comparison with a map produced at the end of the experience. It was noted that the students demonstrated basic mapping skills and simplistic directional language. The senior students worked collaboratively with the teachers to develop a rubric and set their learning goals for the project. A reflective journal was introduced as a tool for tracking student thinking and learning after each task or session. A collection of work samples, video footage and recorded conversations served as formative assessment and supported future planning.

The summative assessment task required the senior students to produce an accurate scale map of the school grounds and directions which led to one of the six hidden geocaches within the school grounds. The use of the maps by a Grade 5 and 6 peer student and Prep buddy provided peer assessment of the effectiveness of the maps to find the geocache. The Grade 5 and 6 students completed a self-assessment questionnaire about their own map and discussed this evaluation in a one-to-one conference with their teacher. The prep students' final assessment was to reflect verbally and present their experience to the larger group.

The teachers witnessed the students' appreciation for mathematics as a dynamic discipline that assisted in their interpretation and making sense of a wide range of experiences, as illustrated in an interview with a Grade Six child, Edward, who stated, '*With school work it's not as fun as geocaching ... it's all maths, writing and work. With geocaching you get to find things, learn more things like compass and directions.*'

The students voiced a newly found appreciation for mathematics as they connected classroom tasks with real life experiences. Geocaching provided the opportunity for the students to become actively involved in rich mathematical tasks. The students could identify a real purpose to learning to draw maps to scale, writing clear concise directions, learning to use a compass, and reading coordinates. The children's reflections highlighted growth and refinement of their mathematical knowledge and an appreciation for the social dynamics of learning, as illustrated in the following except from one student's reflective journal.

The compass points are the main part of a compass. They are also called "Cardinal Points". I learned that a compass is very hard to use...and find something in a snap. I learned that geocaching is not only a way to spend weekends but it requires logic and maths. You also need a sense of direction and self control so you don't get lost. I learned how to use a compass properly by working with my partner and communicating with her. – Rosy, May 5th 2010

The teachers witnessed an increased understanding of the students place in space and an improvement in the appropriate use of directional language. The Prep students began to produce more detailed maps; adding road signs, bridges and compass points. A Prep student, Thomas, was able to use the following directional language to explain his map, 'the bridge goes over the pond' and 'the road is crossing the train tracks'.

A month into the project and the Prep students extended their inquiry about the compass and compass points to an exploration of the north and south poles. The senior students have been able to transfer and extend their understandings from the Geocaching Project to other mathematical tasks including making the links to angles, circles and scale when building tribal huts during their current project.

Conclusion

While it was expected that the students would be interested in the Geocaching Program, the level of engagement, excitement and wonder was unprecedented. There was a buzz in the air, a spring in their step and children were pleading with the teachers to be part of the geocaching group. Some students have borrowed the GPSs to find and hide their own geocaches after school and on the weekends with their friends and family. One student, who was well known for his many years of disengagement in all aspects of schooling, was eager to share his geocaching stories with the school staff and students each day. He, like many of the senior students, was also extremely keen to assist the Prep students with how to read a compass and eagerly guided them during the geocaching trips. Figure 4 shows the cross-aged tutoring taking place during the geocaching project.

Figure 4 - Cross-aged tutoring

Another unexpected and heart- warming result was the level of family involvement in geocaching. A number of school families, like Rosy's, were creating their own geocaching teams and spend weekends treasure hunting. Geocaching is a relatively inexpensive, family friendly activity. In less than three months one of the families from the school has found over 70 geocaches and is still going strong.

Geocaching offers a student-centred, technologically rich adventure which ignites enthusiasm and engagement in mathematics for students of all ages. All children participated and felt success during this innovative and authentic program. The geocaching experience highlighted to the educators involved the need to ensure that tasks are relevant and real for students to generate genuine engagement and excitement in mathematics.

Websites

www.geocaching.com

References

Ainley, M. (2004). *What do we know about student motivation and engagement?* Paper presented at the the annual meeting of the Australian Association for Research in Education.

Australian Curriculum Assessment and Reporting Authority. (2010). The Australian Curriculum (draft). Retrieved 23/3/10, from http://www.acara.edu.au

Christie, A. (2007). Using GPS and geocaching engages, empowers and enlightens middle school teachers and students. *Meridian Middle School Computer Technologies Journal, 10*(1), 1-15.

Gerofsky, S. (1996). A Linguistic and Narrative View of Word Problems in Mathematics Education. *For the Learning of Mathematics, 16*(2), 36-45.

Nunes, T., Schliemann, A. D., & Carrere, D. W. (1993). *Street mathematics and school mathematics.* Cambridge: Cambridge University Press

Piaget, J. (1937). *La construction du réel chez l'enfant. [The construction of reality in the child].* Neuchâtel: Delachaux et Niestlé.

Sparrow, L. (2008). Real and Relevant Mathematics: Is It Realistic in the Classroom? *Australian Primary Mathematics Classroom, 13*(2), 4-8.

The Department of Education and Early Childhood Development. (2010). Mathematics Online Interview. Retrieved July 14th 2010, from http://www.education.vic.gov.au/studentlearning/teachingresources/maths/interview/moi.htm

A ROAD WELL TRAVELLED

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In this paper, observations are reported on various methods used to tackle one specific problem by students from different year levels. Reflections are made on the similarities and the differences between the methods. The comparisons of the methods are considered in relation to the accuracy of the answer and also in terms of their appropriateness to the year level of the students involved. Collectively, the methods provide an opportunity for students to develop a varied approach to problem solving.

Beginning with a problem

I presented a problem to a group of twelve Year 8 students. The problem related to a road circuit. Four towns A, B, C and D were already connected by roads as shown in Figure 1.

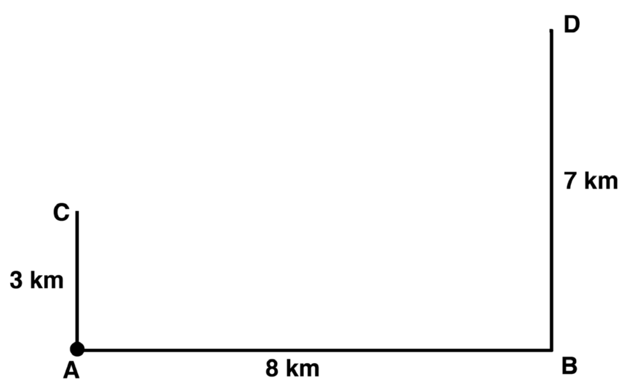


Figure 1. Diagram of roads connecting the four towns.

To help with traffic flow a roundabout was to be placed somewhere along the road between A and B. Connecting roads were to be constructed from the roundabout to towns C and D. But where should the roundabout be placed in order to minimise the cost of building the extra roads?

Using the geometry application in the ClassPad calculator, the students, with my guidance, constructed a scale diagram of the road circuit as shown in Figure 2. To begin with they placed the roundabout at some point near the centre of the road connecting towns A and B. The calculator was used to display the perimeter of the circuit for various positions of the roundabout along the road from A to B. The students were able to display two measurements on the calculator screen. The first measurement was the perimeter of the circuit and the second measurement was the distance from town A to the roundabout at E.

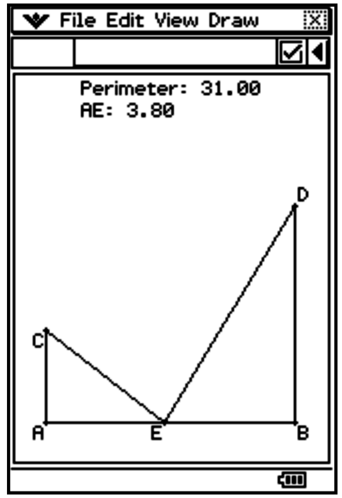


Figure 2. A scale diagram of the road circuit on the ClassPad.

The students then ran an animation in which the roundabout was placed at regular intervals along the road from A to B. The measurements displayed on the screen were automatically updated. By watching the animation the students could see approximately where the roundabout should be placed in order to minimise the perimeter of the road circuit. The animation is illustrated in the screenshots in Figure 3.

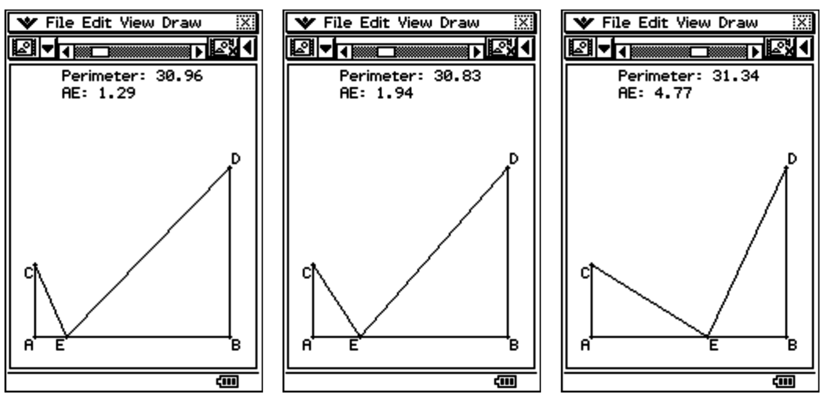
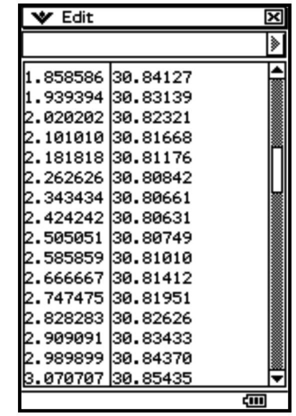


Figure 3. Screenshots of the animation.

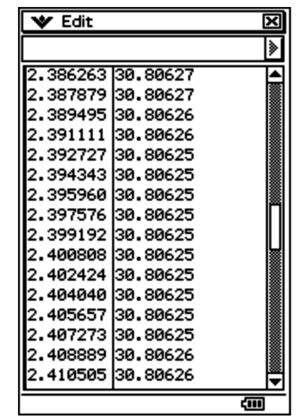
After watching the animation the students were able to produce a table of results from it. The table showed the distance from town A to the roundabout and the perimeter of the road circuit. In this way, the students now had a chance to connect more directly with the numbers involved. They could determine from the table (see Figure 4.) that the best answer would be obtained if the roundabout was placed at approximately 2.4 km from A.



1.858586	30.84127
1.939394	30.83139
2.020202	30.82321
2.101010	30.81668
2.181818	30.81176
2.262626	30.80842
2.343434	30.80661
2.424242	30.80631
2.505051	30.80749
2.585859	30.81010
2.666667	30.81412
2.747475	30.81951
2.828283	30.82626
2.909091	30.83433
2.989899	30.84370
3.070707	30.85435

Figure 4. A screenshot of the results.

With some further guidance, they were then able to adjust the animation so that it collected measurements from around this region in order to gain a more accurate result. The technology in this sense satisfied the students' natural inclination to “zoom in” on the numbers. In this way the students obtained an answer for the minimum perimeter at 30.81 km when the roundabout was placed at 2.4 km along the road from A to B as shown below in Figure 5.



2.386263	30.80627
2.387879	30.80627
2.389495	30.80626
2.391111	30.80626
2.392727	30.80625
2.394343	30.80625
2.395960	30.80625
2.397576	30.80625
2.399192	30.80625
2.400808	30.80625
2.402424	30.80625
2.404040	30.80625
2.405657	30.80625
2.407273	30.80625
2.408889	30.80626
2.410505	30.80626

Figure 5. A more accurate answer is obtained.

A solution using similar triangles

When I presented the same problem to a class of Year 10 students they used the same method as the Year 8 students. In addition, however, they were able to use another completely different method for solving the problem. By interacting with the diagram on the screen the students were able to produce an adaptation of it as shown below. (The students were shown how to set the diagram so that the fixed lengths, the right angles and the slope of the line from A to B would remain unchanged). The problem now focused on minimising the distance from C through E to D. It was clear to them that the shortest distance would be found when C, E and D all lay on a straight line as shown in Figure 6. This was intuitively obvious to the students given their existing knowledge that the shortest distance between two points (C and D in this case) is a straight line.

The problem then became a useful example of the use of similar triangles which was demonstrated to the students. A tablet computer connected to a digital projector was used for this purpose as shown in Figure 7.

Hence, an exact answer to the problem emerged by deductive means, which corroborated the previous answer obtained by repeated measurements.

It is worth reflecting on how the solution obtained by the Year 8 students using a table of values compares with the solution found by using similar triangles. The table of values method relies essentially on a trial and error approach and cannot with certainty produce an exact answer. In contrast with this, the similar triangles method is exact and unequivocal. The methods are conceptually different, however, and it is wiser to allow the Year 8 students the opportunity to adopt an approach involving numbers before attempting a more abstract method.

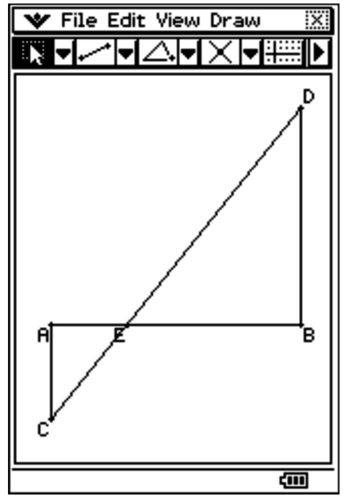


Figure 6. The shortest distance between two points is a straight line.

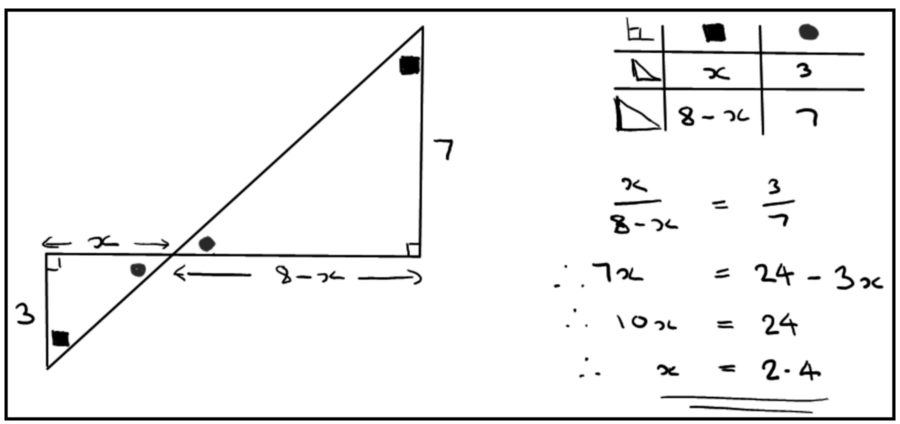


Figure 7. A solution using similar triangles is demonstrated to the students.

A solution using Pythagoras' Theorem

The problem was presented once more, this time to a Year 11 class. The method of repeated measurements with the aid of the animation was carried out as before. In addition, however, the students were guided towards another approach to the problem that entailed

the use of Pythagoras' Theorem. In the course of a class discussion, a labelled diagram was developed by the students as shown in Figure 8. The diagram supported a method of solving the problem which combined symbolic manipulation and graphing.

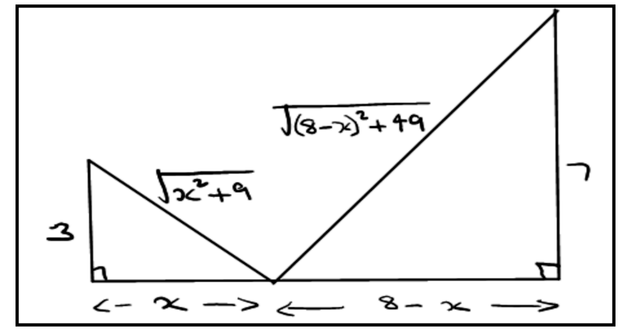


Figure 8. Using Pythagoras' Theorem.

By entering a formula for the perimeter into the graphing calculator application of the ClassPad, the students were able to produce a graph from which the best position for the roundabout could be identified along with the minimum perimeter of the road circuit as shown in Figure 9.

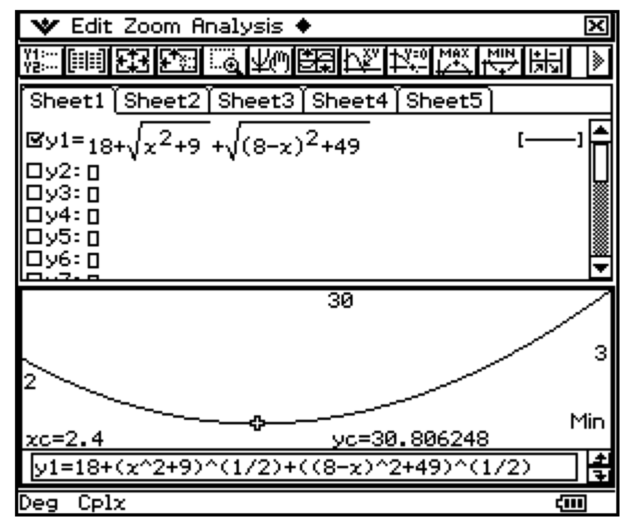


Figure 9. Year 11 students find a graphical solution.

This method using Pythagoras' Theorem involves a good deal of algebraic manipulation which was not present in the method that the Year 8 students employed. The Year 8 students produced an answer from their tables that was "probably exact". It may appear that the Year 11 students were not producing an exact answer either since they were reading their answer from a graph. The important distinction that needs to be made, however, is that the Year 11 students were able to derive an exact function on which to base an answer. This method could therefore be considered to be more robust than the method used by the Year 8s. The Year 11 students used technology to convert the function to a table and then to a graph but this should not overshadow the underlying rigour of their method.

A solution using differential calculus

Finally, the problem was given to a Year 12 class. With characteristic zeal they strove to establish an algebraic relationship and use differential calculus to solve the problem. They first obtained a formula for the perimeter in terms of the distance from A to the roundabout as shown below:

$$P = \sqrt{x^2 + 9} + \sqrt{((8 - x)^2) + 49} + 18.$$

Using the computer algebra system (CAS) in the ClassPad, they found the derivative of the perimeter function and equated it to zero as shown in Figure 10.

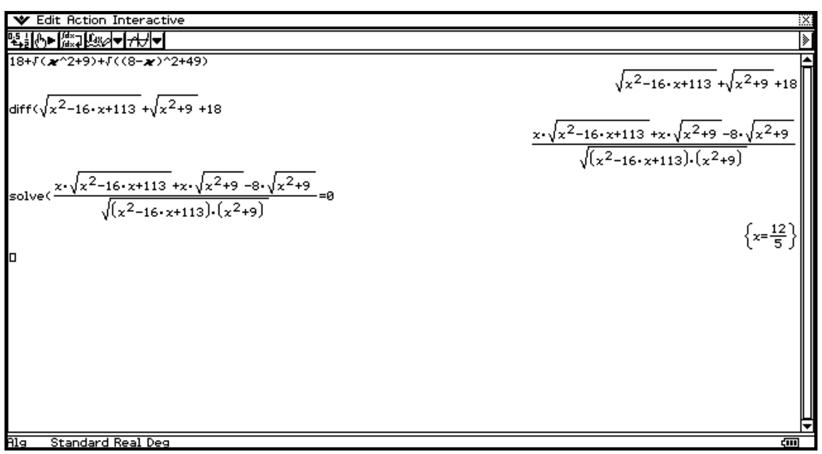


Figure 10. Year 12 students use differential calculus.

This produced the exact answer of $\frac{12}{5}$ or 2.4 in agreement with the other answers. With the aid of the CAS, therefore, the students obtained a solution that came out of a process of symbolic manipulation. The use of the CAS in this case allowed students to focus on the problem rather that what would have been a lengthy by hand manipulation (Broline 2007). Students appreciate the use of a CAS in this situation (Pierce 2001). As with the method using similar triangles, the approach using differential calculus produced an exact answer.

Review of the methods used

The various methods that were used can be considered in terms of the levels of mathematical knowledge that they require in relation to the year levels of the students. They form a hierarchy in the sense that the older students have access to more and more sophisticated techniques than the younger ones. The methods can also be categorised according to the nature of the solutions obtained. The answer arrived at by the use of similar triangles was exact and so too was the one found using calculus. The answer found from the repeated measurements is not perfectly accurate, nor is the one found by graphing. It should be noted, however, that the similar triangles method had the highest level of intrinsic merit since it produced an exact answer using deduction. The interactive nature of the technology supported this method by allowing an adapted representation of the problem to be displayed.

We can assist students through the process of acquiring and integrating knowledge by allowing them to experience varied representations and solutions to the same problem, some numeric, some visual and some symbolic. Often this will involve careful sequencing of approaches, for example working with numbers prior to graphing and leaving a purely algebraic approach until later. Some guidance from the teacher will be required initially although, as in the examples described above, the students are actively involved and not just passive recipients of an entirely teacher led exposition (Lapp 2009). Apart from the benefits in relation to gaining knowledge, however, the use of different methods will also broaden the students' perceptions of what it means to do mathematical work. Hopefully, they will gain an appreciation of the fact that problems can often be solved in more than one way and that some methods of solving problems can be more highly regarded than others.

References

Broline, D. (2007). *"Examples of Using a Computer Algebra System in Teaching Algebra."* Primus : Problems, Resources, and Issues in Mathematics Undergraduate Studies **17**(3): 256-267.

Lapp, D. (2009). *"Dynamically Connected Representations: A Powerful Tool for the Teaching and Learning of Mathematics."* International Journal for Technology in Mathematics Education, The **16**(1): 37-44.

Pierce, R. (2001). *"Observations on Students' Responses To Learning in a CAS Environment."* MATHEMATICS EDUCATION RESEARCH JOURNAL **13**(1): 28-46.

AN EFFECTIVE NUMERACY PROGRAM FOR THE MIDDLE YEARS.

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Sunshine College

An holistic approach to improving student numeracy through the implemention of an effective Mathematics program for all middle years students, incorporating problem solving; improving mathematical literacy; information and communication technology and the scaffolding of numeracy concepts in a fully differentiated classroom.

Introduction

Sunshine College is a multi-campus Government secondary school located in the Western Metropolitan Region (WMR) of Melbourne. It was formed in 1991, following the reorganisation of six secondary schools and has an enrolment of approximately 1000 students. It is positioned across four sites and is made up of three junior campuses including a deaf facility and one senior campus. It is a culturally diverse school with more than 50 language backgrounds. The population, in general, suffers a high degree of disadvantage and a low socio-economic position. In excess of 60% of families are in receipt of Educational Maintenance Allowance (EMA) (according to the school's Annual Report 2009).

In general, the majority of Mathematics classes at Sunshine College are teacher directed with the classroom teacher delivering the lesson from the front of the room. The teacher will then complete a number of worked examples on the board, which the students copy into their workbooks followed by various exercises from the Mathematics textbook. Classes rarely use concrete manipulatives; students are expected to work individually; assessment is summative; and the opportunity for modification is limited with weaker students expected to complete fewer examples than the more competent students. On each of the junior sites all students receive four fifty-minute periods of Mathematics instruction per week.

In 2008 and after several years of little or no improvement in data (AIM & VCE), and the placement of several numeracy coaches from the WMR, Yvonne Reilly and Jodie Parsons began to develop an alternative numeracy program.

The Whole School Numeracy Program

The pedagogy of the revised whole school numeracy program is purposeful. The curriculum is derived from the VELS Mathematics continuum and each unit of work is based on understanding the Victorian Essential Learning Standards (VELS) levels of our students as determined by On-Demand data.

This means that in any class we are planning units of work in any given topic area for students from VELS level 2 to VELS level 6, and we are obligated to produce lessons that are differentiated and where every student has not only a point of access but also a stake in the class. We have observed that when our lower ability students perceive of that they are completing the same work as their higher ability peers, positive reinforcement occurs. This is a phenomenon extensively documented by Boaler in her studies of Mathematics instruction in the UK (2001). This approach to planning is a fundamental component of our revised program, where the curriculum focuses on teaching at each individual student's level, providing truly individual learning outcomes.

The delivery of the numeracy program is varied. At times the curriculum is delivered as a whole class investigation with all students generating their own examples and at other times the curriculum is delivered using a range of tools such as open ended tasks, group work, Information Communication Technology (ICT) and the production of artefacts. To accommodate the successful implementation of the numeracy program we have developed a fortnightly schedule as described in Table 1.

Table. 1 Fortnightly schedule for the delivery of a whole school numeracy program

	Lesson 1	Lesson 2	Lesson 3	Lesson 4
Week 1	Scaffolding numeracy	Differentiated lesson	Differentiated lesson	Reciprocal teaching
Week 2	Scaffolding numeracy	Differentiated lesson	Differentiated lesson	ICT

The whole school numeracy program is composed of four main areas:

- Scaffolding numeracy,
- Differentiated curriculum,
- Reciprocal teaching
- Information computer technology.

Scaffolding numeracy

One fifty-minute period per week is dedicated to addressing mathematical misconceptions using the Scaffolding Numeracy in the Middle Years (SNMY) program as developed at RMIT University by Professor Di Siemons and her team (http://www.education.vic.gov.au/studentlearning/teachingresources/maths/snmy/projbkgd.htm).

This body of work addresses the big ideas in the development of mathematical understanding. It moves students from additive thinking to the more efficient multiplicative strategies and beyond to proportional reasoning.

Whilst the program was designed primarily for students in Years 5 and 6, many of our students are entering secondary school well below the expected level. Of the 2010 cohort at our school, 14% were at or above expected level; 36% one year below; 32% two years below, 7% three years below and 11% four or more years below, so the scaffolding program is at an appropriate intellectual level for our students. Alongside the scaffolding lessons we also conduct 'normal' Mathematics classes in a truly differentiated way.

Differentiated Curriculum

Two fifty-minute periods of Mathematics instruction per week are dedicated to the delivery of a differentiated curriculum. The structure of each lesson is based on the model described by Rob Vingerhoets for the WMR. Each lesson begins with a warm up activity followed by a five-minute teacher introduction to a student centred activity. The student centred activity, often a group task, provides a rich learning experience and is designed for each student to discover learning for themselves. It also provides an opportunity for the classroom teacher to walk around the room questioning students and teaching explicitly at the point of need. At the conclusion of every Mathematics period, students are required to reflect on their learning either individually or as part of a group.

This format, as described by Rob Vingerhoets (WMR Professional Learning throughout 2009 and 2010) fits perfectly with Kalantzis and Cope's (2005) belief that experiential learning is informal and that "the best of formal learning accounts for and integrates informal learning into its patterns and routines"(pp38). It is also in line with WMR best teaching practice.

Reciprocal Teaching

One fifty-minute period per fortnight is dedicated to reciprocal teaching. Our Reciprocal Teaching for Mathematics strategy, although based on the model proposed by Palinscar and Brown (1984), has a number of key adjustments described fully by Reilly, Parsons and Bortolot (2009). This revised Reciprocal Teaching is a strategy for improving mathematical literacy where students work in small groups to bring meaning to the mathematical text of written problems. It comprises of four stages; predicting, clarifying, solving and summarising.

Information Communication Technology

One fifty-minute period of instruction per fortnight is dedicated to ICT, although ICT is used wherever possible, this lesson ensures our students have the opportunity to develop competencies for the digital nature of their future.

Assessment

At the beginning of each unit, students are assessed for their pre-existing knowledge of the curriculum, the lessons and the learning opportunities are then planned accordingly. Each student is given a self assessment rubric which provides them with an opportunity to demonstrate their pre-existing knowledge and to identify areas of deficit. The self-assessment rubric which we call the 'Happy Face' sheet asks students to tick their level of confidence with a variety of learning criteria. Their ticks are placed on a continuum from 'I don't understand this yet' to 'I've got it! I could teach someone else'. Any student who selects the option: 'I've got it! I could teach someone else' must provide some evidence that this is the case in the box provided.

The self-assessment rubric has a two-fold benefit; it not only provides us with evidence of genuine or lack of understanding but also allows the students to see a progression of their learning, as the rubric is filled in at both the beginning and the end of the unit. This information is then used to inform our differentiation, planning, and grouping.

This approach we believe is a fundamental component of the revised curriculum, for historically, it has not been unusual for teachers to teach according to a student's year level, however, this curriculum focuses on teaching at each individual student's level; providing truly individual learning outcomes.

At the beginning of each unit students are provided with a task sheet which informs them of the goals and standards they are expected to achieve, and the criteria they will be assessed against. This task sheet helps students reflect on their thinking, to plan their work, to monitor their understanding, and evaluate their progress. The overall goal is to teach students to self-manage and self-monitor their learning. The task sheet allows students to select (with guidance and encouragement from the classroom teacher) from a variety of activities to develop their learning and understanding. Each activity on the task sheet is scaffolded and activities are planned from VELS levels 2 to 6.

Topic tests are standard practice in secondary Mathematics programs; however, from our experience we have noted that it is not unusual for students who are operating at four years below the expected level to achieve 70% or more in a class 'topic' test. The question is then, what are these tests actually testing? To compound the detrimental effect of these tests, when they are reported on these tests are both arbitrary and misleading for parents, who would assume that their children are achieving a better than average score.

The summative information provided by the On-Demand testing of our students, although useful for generating "ball-park" information on our students, it does not provide evidence about which specific learning outcomes have been achieved, therefore additional means of assessment are employed, such as Scaffolding Numeracy testing options; student self-assessment with the Happy Face sheet; diagnostic tests from the Department of Education and Early Childhood Development (DEECD); formative assessment tasks. Our assessment is ongoing and purposeful. Students are regularly encouraged to justify their answers by explaining procedures to peers and with opportunities for assessment by interview.

There have been a number of other aspects of the revised numeracy program which have proved to be beneficial:-

A framework for team teaching;

- Opportunities for students to separate into homogenous groups;
- Pairing up with a higher skilled peer for problem solving activities.

The revised numeracy program also provides staff with advantageous outcomes:-

- Shared planning, ideas and resources;
- Personal and professional development;
- Joint reflection and moderation of work;
- Mentoring, modelling and coaching.

The benefits of all of the above translate into better quality teaching and ultimately improved student learning (Stephens, 2009).

The implementation of a cross-campus curriculum for the teaching of Mathematics in years 7-10 have achieved the following outcomes:-

- Improved student learning in Mathematics as measured by NAPLAN and On-Demand assessments.
- Improved student engagement (student survey data)
- Consistency within the college regarding the curriculum, its delivery and enhancing transition to senior campus.
- An approach that is consistent with the WMR direction as described in the Blueprint For Schools.
- Preparation for anticipated merger of the three junior campuses.
- Shared planning and preparation thus allowing teachers to work smarter, not harder.
- Whole school analysis of data to inform the planning of effective learning.
- Consistency in report writing.

The Hurdles

As the uptake of the program by college staff is central to our goal of improving whole school data, it was important for us to manage the change process as effectively as possible. Our goal was to "pursue a successful program of .... organisational change" (Sparrow and Knight 2006, pp.xiv) and not just change for the sake of it or change the practice of only a few individuals as this would not have a significant effect on our data. To this end, we felt that we had to:-

Convince our staff of the need to change;

1. Show that our program had been tested and refined;
2. Demonstrate that the data collected was impartial and rigorous and not solely anecdotal;
3. Prove that each aspect of the program had a legitimate reason for inclusion and that nothing was changed just for change's sake; and
4. That we were in it with them, teaching similar students.

To this end we have endeavoured to 'prove' that the changes we proposed to teaching numeracy really work.

Essential components of the whole school numeracy program

Delivery of a fully differentiated curriculum with a team of teachers can only occur if the lessons are planned and delivered with contributions from all members of the team. This ensures that the students receive the best possible learning outcomes. All teachers teaching the unit take ownership of the lessons and ensure that they are fully informed of the expected outcomes for each lesson. Additionally, team planning provides consistency, transparency of practice and accountability. Team planning also provides opportunity for reflection however, it is essential that all involved are provided with an adequate allocation of planning time and can agree on a common set of goals.

Conclusion

Over the past 18 months, On-Demand data has been collected from two populations of students, one that had been taught using the revised whole school numeracy program, the other group taught using traditional teacher centred practices. This data indicated that students taught using the revised whole school numeracy program had improved, on average, by 0.3 VELS during the year (On-Demand General Adaptive Test), whilst the students who have been taught in the traditional manner had registered an average improvement of only 0.1 VELS level per year. This means that whilst our students do not yet attain the state average yearly improvement of 0.5 VELS, they are improving three times more than the students at our school who are not in the program.

This three-fold increase in student achievement was recognised by Monash University audit of the School's Strategic Implementation Plan. They recommended the program be extended to all junior campuses. (Auditor's Report on Sunshine College Strategic Implementation Plan, 2006-2009).

Additionally, the Ardeer Campus Year 9 students are the first students to complete two full years of the Numeracy Program and we anticipate that the 2010 NAPLAN data from these students will be indicative of the efficacy of the program.

References

Boaler, J. (2001). Mathematical Modelling and New Theories of Learning Teaching Mathematics and its Applications; Sep 2001; 20, 3; *ProQuest Education Journals* pp. 121.

Brown, V. L., Hammill, D. D. & Wiederholt, J. L. (2008). *Test of reading comprehension*.

Council of Australian Governments. National numeracy review: final report, commissioned by the Human Capital Working Group, COAG, Commonwealth of Australia, Canberra. Available at: http://www.coag.gov.au/reports/index.cfm

Christensen, C. A. & Wauchope, M. (2005). *Whole school literacy: Using research to create programs that build Universal high levels of literate competence.* University of Queensland.

Geary, D.C. (1994). *Children's mathematical development: research & practical applications.* Washington DC, American Psychological Association 1994.

Geist, E. (2009). *Children are born mathematicians: supporting mathematical development, birth to age 8.* Merrill/Pearson xvi, p398.

Gifford, M. & Gore, S. (2008). *The Effects of Focused Academic Vocabulary Instruction on Underperforming Math Students. Association for Supervision and Curriculum Development Report.*

Palincsar, A.S., & Brown, A.L. (1984). Reciprocal teaching of comprehension-fostering and comprehension-monitoring activities. *Cognition and Instruction,* 1, 117-175.

Goleman, D. (1996). "On emotional intelligence," Educational leadership, v.54, n.1, 6-10.

Kalanzis, M. & Cope, B. (2005). *Learning by Design*, p38, Common Ground.

Marzano, R., & Pickering, D. (2005). *Building Academic Vocabulary: teachers' manual. Alexandria*, VA: Association for Supervision and Curriculum Development.

Reilly, Y., Parsons, J. & Bortolot, E., (2009). *Reciprocal teaching in mathematics. Mathematics of prime importance.* MAV Annual Conference 2009.

Stephens, M. (2009). *Numeracy in practice: teaching, learning and using mathematics.* Paper No. 18 June 2009.

Sparrow, T., & Knight, A. (2006). *Applied Emotional Intelligence: The importance of attitudes in developing emotional intelligence.* pp.xiv.

Vingerhoets, R. (1990). *The Maths Workshop*. Eleanor Curtain.

Vingerhoets, R. (2001). *Maths on the go 1*. Macmillan.

Vingerhoets, R. (2006). *Maths on the go* 2. Macmillan.

http://www.education.vic.gov.au/studentlearning/teachingresources/maths/mathscontinuum/default.htm

http://www.education.vic.gov.au/studentlearning/teachingresources/maths/snmy/projbkgd.htm

PERFORMANCE OF LOW ATTAINERS FROM SINGAPORE IN NUMERACY

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This paper examines the performance of 466 Primary 4 (Year 4) low attainers from Singapore in numeracy. The pupils are identified as low attainers by their respective mathematics teachers. Their responses and performance in a series of three numeracy benchmark tests are discussed under the categories; mathematical skills, number concepts and word problems. Attempts are also made to examine the difficulties experienced by these pupils.

Introduction

This paper is a part of a major study on low attainers in mathematics. In the main study (Kaur & Sudarshan, 2010), different instruments are used to gather data to build an intensive profile of the cognitive, affective and family-related correlates for a group of Primary 4 (Year 4) pupils in Singapore. They are considered low attainers by their teachers based on the school assessments. This paper reports on the differences in their performance in selected components of numerical proficiency and attempts to answer the following questions:

Do these pupils show consistently poor performance across the different components of numerical proficiency?

What are some mathematical learning difficulties pupils have that prevent them from achieving the required competency in arithmetic for their grade level?

Definition of low attainers in mathematics

Researchers (e.g., Geary, 2004) have used various discrete aspects of numerical proficiency such as strategic counting, retrieval of basic number facts, numeral recognition to predict future success in mathematics. In Singapore, a School Readiness Test is used to identify Primary1 pupils who are at risk of serious reading difficulties and/or mathematics difficulties. Instructional programmes such as Learning Support Programme for both English and Mathematics are tailored for these pupils at Primary 1 and 2 levels. The programme is not extended beyond Primary 2. Therefore, pupils who continue to fail in mathematics beyond Primary 2 do not have access to a structured programme for help and therefore are at risk henceforth. According to the syllabus (Curriculum Planning and Developement Division, 2006), Primary 4 pupils are expected to have mastery of the following at the beginning of their school year:

- Representation of numbers to 10 000
- Magnitude comparison of numbers up to 10 000
- Place value from ones to thousands
- The four operations
- Word problems

In this study, we studied Primary 4 low attainers in mathematics who have many gaps in their mathematical knowledge and have not developed required numerical proficiency in these topics.

Characteristics of low attainers in mathematics

Children with mathematics difficulties have been reported to have incomplete mastery of basic number facts, incompetent arithmetic computations, difficulty transferring knowledge, making connections, and incomplete understanding of the language of mathematics, particularly in interpreting word problem sentence construction (e.g., Gracia, Jimenez & Hess 2006). Bryant, Bryant & Hammill (2000) reported that the top seven most common mathematics difficulties as ranked by a group of special education professionals are as follow:

1. difficulty with word problems
2. difficulty with multi-step problems
3. difficulty with the language of mathematics
4. failure to verify answers and settle for the first answer

5. lack of automaticity with number facts
6. take a long time to complete calculation
7. make 'borrowing' (i.e. regrouping, renaming) errors

Solving word problems and computation represent two distinct domains of mathematical cognition in low attainers in mathematics (Fuchs at al., 2008). Pupils identified with mathematics difficulties in the schools may not show consistently poor performance in both areas.

Methods

Sample

Nine neighbourhood schools across Singapore participated in the main study. A total of 490 Primary 4 pupils were first identified by their teachers as low attainers in mathematics. These are pupils at risk of failing their Primary School Leaving Examination in Primary 6. Two pupils were later excluded from the study when it was found that they could neither write their names properly nor answer any written mathematics test item correctly. Among the remaining 488 pupils, only 466 completed the tests on whole numbers.

Instruments

Three written tests on whole numbers were used to measure whether the pupils reached the expected proficiency level in arithmetic at the end of Primary 3. Test 1 assesses pupils' conceptual understanding of whole numbers including numeral recognition, number comparisons and place value. Test 2 assesses pupils' understanding of the four operations and their procedural/computational knowledge. Test 3 assesses pupils' ability to solve up to 2-step word problems.

Data collection

The tests were administered to the pupils, in three separate sittings, over a span of three weeks by their teachers during instructional time. As the tests assessed mathematical knowledge that the pupils are expected to know, the pupils were not informed of the test in advance and hence did not make special preparation for the tests.

Results and Discussion

The overall proficiency level

Table 1 shows the mean and standard deviation of the test scores. Pupils performed relatively well in the test on number operations and rather badly in the test on word problems. Hence it seems that many low attainers in the study are deficit in problem solving skill but not the computation skill.

Tests	Maximum score possible	mean	SD
Number concepts	**30**	16.79	4.69
Number operations	29	21.94	4.09
Word problems	9	3.38	2.42

Table 1: The means and standard deviations of the test scores for the tests on whole numbers (n= 466)

Numerical concepts

Only test items with facility indices less than 0.80, as shown in Table 2, will be discussed.

Item	Item description	Facility Index
4a	Place value	0.6588
5a	Place value	0.3991
8	Number comparisons (form smallest 4-digit number with given digits)	0.4592
11	Number comparisons (State a number in a given range)	0.3541
12	Number comparisons (Order 4 given numbers)	0.7747
15a	Find missing term in number pattern State rule (add five)	0.6052 0.3949
15b	Find missing term in number pattern State rule (multiply by 2)	0.2189 0.1159
15c	Find missing term in number pattern State rule (add 100)	0.6931 0.5150
16	Identify a number closest in value to a given 3-digit number. Identify a number closest in value to a given 4-digit number.	0.6116 0.3755
17	State the missing numbers on a given number line. (Scale 1: 1) State the missing numbers on a given number line. (Scale 1: 10)	0.7554 0.0708

Table 2: Facility indices of selected items in Test 1 (number concepts)

Place value concept

Twenty four percent of pupils (n=116) wrote variations of the wrongly spelt 'tens' (e.g. tans, tends). Similar errors are found for Items 4b and 5a. Pupils seemed able to identify the correct place value of the digit in a number but have difficulty spelling the required word. Pupils also appeared confused as to when to put down, as answer, the place value or the value. This stemmed from a poor understanding of the oral and written form of a number and the base-ten language.

Number comparison

This concept is discussed through three groups of items. In the first group (Item 8), pupils were required to apply their knowledge of place value concept to form the smallest 4-digit number. The most common mistake made was forming a 3-digit number (i.e. 459), omitting the given digit zero. These pupils either do not know the meaning of '4-digit' or the function of zero. Comparing these with the responses of Item 9 where pupils were required to form the largest 4-digit number, the difficulty seemed to be the lack of understanding of the word '4-digit'.

In the second group (Item 11), pupils were required to apply their knowledge of number sequencing to pick a number from a given range. Twenty three percent (107 pupils) gave the difference of the numbers at both ends of the range. Plausible reasons are: first, they did not understand the requirement of the item posed, second, they did not accept a possibility of more than one answer or a range of answers and third, they were not able to handle the many parameters given in the item.

In the third group (Item 12&13), pupils were required to arrange four 4-digit numbers in increasing or decreasing order. Two common mistakes found were arranging the numbers in the reverse order stated (2.58%) and incorrect order of the last two numbers only (6.87%).

Pupils in the latter group knew how to order the numbers but found it overwhelming and confusing to order the 4 given numbers made up of same four digits in different place value. Their working memory capacity may be limited thus hindering their ability to remember simultaneously all the four numbers that looked the same. To help them keep track of the information, some pupils were observed to 'cancel' the number already considered.

Number Patterns

Pupils were more competent in identify number patterns involving addition (64.92%) than multiplication (21.89%). Most pupils (47.21%) regarded the pattern in Item 15b as an arithmetic series in multiples of 2 or plus twos, rather than a geometric series (multiply by 2). This shows pupils' preference in handling addition rather than multiplication. Pupils found it difficult to explain the rule of each pattern. They lack communication skill. However, for Item 15a, 12.45% gave 'minus 5' as the rule rather than 'add 5' which make sense if we look at the pattern backwards.

Number closest to value

In Item 16a and 16b, pupils were required to identify from four given options, one that is closest to 644 and 2097 respectively. From pupils' errors for Item 16a, 24.68% of the pupils chose either 650 or 600 as the answer while for Item 16b, 31.55% of the pupils chose 2100 as the answer. It was noted that the appearance sequence of all the chosen options stated above was either first or second. Apparently, these pupils failed to verify their answers and opted for one of the first two options.

Representation on number line

For Item 17a, 75.54% of the pupils were able to insert the correct missing number where one unit represents one. But for Item 17b, where one unit represents ten, only 7.08% of the pupils could state the correct missing number. These pupils (61.80%) mistook each unit as one and gave 2782 and 2785 respectively for each box. They did not take into account all information presented and assumed the same rule applied for both parts of the item. They had either overlooked the information presented later or failed to look back to make sense of their responses against the given information.

Number operations

There are two parts to Test 2. Item 1a to 1g consist of problem situations which required pupils to identify, from four options, the correct number sentence for each situation. Items

2 to 5 consist of column addition & subtraction and recall of simple multiplication facts (see Table 3).

Item	Item description	Facility index
1a	Join, initial quantity unknown	0.5687
1b	Equal groups, total unknown	0.7017
1c	Equal groups, number of groups unknown	0.4871
1d	Part-part-whole, part unknown	0.3455
1e	Equal groups, number of groups unknown	0.5017
1f	Comparison, difference unknown	0.4034
1g	Equal group, total unknown	0.5236
4c	3 digit number subtract 3 digit number, renaming tens	0.7854
4d	3 digit number subtract 3 digit number, renaming ones and tens	0.7811
4e	3 digit number subtract 3 digit number, zero in the tens place of the minuend	0.6738
4f	3 digit number subtract 3 digit number, zero in the ones and tens place of the minuend	0.7511

Table 3: Facility indices of some items on the 4 operations (Test 2)

or items on column addition and subtraction of 3-digit numbers, with or without renaming, the facility indices of the items are greater than 0.8900 except for the items involving column subtraction (see Table 3). The facility indices in Table 3 show that pupils have more difficulties identify the correct operation for a problem situation than executing multi-step arithmetic algorithms. Subtraction with renaming is the only algorithm that yielded poor result. The problem worsened with zeros in the minuends (see Fig. 1). In both cases, pupils knew they were required to rename the zero in the minuend. But they calculated incorrectly because they had incomplete knowledge of the procedure involving renaming zero(s).

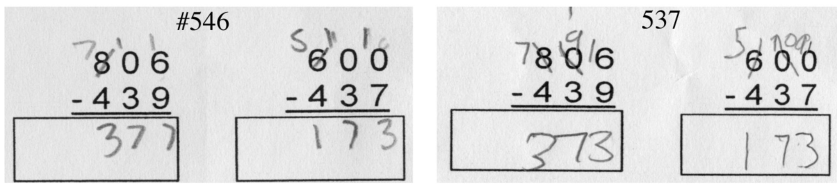

Figure 1: Error in renaming zero in subtraction

Arithmetic word problems

There are altogether two one-step word problems and seven two-steps word problems in Test 3, involving varied part-whole and equal-groups concepts as shown in Table 4. All word problems registered a facility index of less than 0.80 with most two-steps word problems scoring lower than 0.50.

Item	Item description	Facility Indices
1	1-step problem, partition division	0.6931
2	2-step problem, join (total unknown) and separate (result unknown)	0.6137
3	1-step problem, measurement division	0.4399
4	2-step problem, separate (result unknown)	0.5773
5	2-step problem, join (total unknown), equal group (whole unknown)	0.2661
6	2-step problem, equal groups (whole unknown), Comparison (difference unknown)	0.1803
7	2-step problem, equal group (number of groups unknown), part-whole (part unknown)	0.0579
8	2-step problem, separate (result unknown)	0.3498
9	2-step problem, comparison (larger unknown), part-whole (whole unknown)	0.2039

Table 4: Facility indices of items in Test 3.

From the two one-step word problems (Item 1&3), pupils performed better in item involving partition division than involving quotition division. They were able to relate to the former type of division problems because the semantic structure of Item 1 is more familiar to them.

We found some common trends from the errors of pupils in two-step word problems. First, they have difficulties dealing with equal-groups word problems that involve multiplication and division. The preferred operation chosen was addition and/or subtraction. Second, their responses showed an absence of awareness that more than one-step was needed to solve the problem. Third, they tend to pluck out all the given numbers in order of appearance from the word problem and then applied the same operation (either addition or subtraction) for both steps. Other relevant information such as 'earns another', 'spends', 'change' to provide a context in the word problems were ignored. This showed a lack of comprehension of the words they read which is consistent to results in Test 2.

Implications and conclusions

The results reveal valuable information for teachers in the area of instructions, marking scheme and memory strategies.

Instructions

1. Link to be made explicitly between oral and written forms of base-ten language. As both are technical skills, teachers need to make the connection explicitly. Spelling of written base-ten numbers must be taught and reinforced.
2. Explain the concept of zero and its use as a placeholder.
3. Highlight the difference between digits and numbers (e.g. 0,1,3 are three different digits, 013 is a 2-digit number and 130 is a 3-digit number).
4. Include examples of subtraction with multiple renaming involving zeros in the minuends during instruction and practice.
5. Provide a wide range of number patterns that involve arithmetic and geometric order (refer to Item 15).
6. Provide opportunities for pupils to work on problems that entail more than one answer.
7. Emphasize comprehension and representation of word problems when teaching word problems.
8. Create time and opportunity for mathematical communication among pupils by getting them to explain solutions or processes.
9. Provide word problems involving similar operations in varied semantic structure.

Mark Scheme

1. Review mark scheme not to penalize spelling error.
2. Discuss possible alternative solutions in class.

Memory strategies

1. Teach pupils 'cancelling' to help free up working memory space (refer to last paragraph on number comparison).
2. Teach pupils to organize information in word problems such as with a diagram/ chart.

Very often, the problems faced by low attainers are no different from that of their peers. Apart from a seemingly weaker performance in word problems, which is not the focus of this paper, there is no significant cognitive barrier that disabled them from mastering and achieving success in most routine questions posed in schools. Yet their motivation and performance is greatly marred by persistent failure from earlier grades. Given the hierarchical nature of mathematics and the spiraling mathematics curriculum in Singapore, the gap separating them from their peers continued to widen. However, refinement to teachers' instructions, mark scheme and memory strategies as proposed in this paper could come in handy for teachers to help these pupils improve.

References

Curriculum Planning and Development Division (CPDD), (2006). Mathematics Syllabus Primary. Singapore: Ministry of Education

Bryant, D. P., Bryant, B. R., Hammill, D. D. (2000). Characteristic behaviors of students with LD who have teacher-identified math weaknesses. *Journal of learning disabilities,* 33(2), 168-199.

Fuchs, L.S., Fuchs, D., Stuebing, K., Fletcher, J.M., Hamlett, C.L., & Lambert, W.E. (2008). Problem solving and calculation skill: Shared or distinct aspects on mathematical cognition? *Journal of Educational Psychology,* 100 (1), 30-47.

Geary, D.C. (2004). Mathematics and learning disabilities. *Journal of learning Disabilities,* 37(1), 4-15.

Gracia, A.L., Jimenez, J.E. & Hess, S. (2006). Solving arithmetic word problems: An analysis of classification as a function of difficulty in children with and without arithmetic LD. *Journal of Learning Disabilities,* 39(3), 270-281.

Kaur, B. & Sudarsham, A. (2010). An exploratory study of low attainers in primary mathematics (LAPM) - First year report. Singapore: National Institute of Education.

LESSON STUDY – HOW IT COULD WORK FOR YOU

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There is growing worldwide interest in Japanese Lesson Study as a form of professional development, with adaptations of Lesson Study taking place in hundreds of school clusters in the USA, large-scale adoption in the UK, and smaller scale implementation in Australia, and many other countries. This paper describes the typical Japanese structured-problem-solving research lessons that form the basis for Lesson Study, and discusses how they are planned, the role of the teacher, and the use of Lesson Study as a means of professional development.

Introduction

Japanese Lesson Study first came to world-wide attention through Makoto Yoshida's doctoral dissertation (Yoshida, 1999; Fernandez & Yoshida, 2004) and Stigler and Hiebert's (1999) accounts of Lesson Study based on the *Third International Mathematics and Science Study* (TIMSS). By 2004, Lesson Study was taking place in the USA in at least 32 states and 150 lesson study clusters.

In the United Kingdom there has been growing interest in, and government support for, Lesson Study as a powerful form of professional development (see, for example, Department for Children, Schools and Families, 2008).

Adaptations of Japanese Lesson Study are being implemented, often in small ways, in many other countries, including Chile, Indonesia, Malaysia, Mexico, Peru, Philippines, Singapore, Thailand, and Vietnam (APEC-HRD Lesson Study Project, n.d.).

In Australia, there have been a number of small-scale attempts at Lesson Study (see, for example, Hollingsworth & Oliver, 2005; Clarke & Sanders, 2009; Pierce & Stacey, 2009) as well as a larger-scale trial of a modified form of Lesson Study in NSW (see, for example, White & Lim, 2008). However, as Stephens (in APEC-HRD Lesson Study Project, n.d.) points out, "schools need assistance to engage more deeply in the research phase, and to see Lesson Study as part of an ongoing cycle of improvement".

The purpose of this paper is to describe the typical Japanese structured-problem-solving research lessons that form the basis for Lesson Study, discuss how they are planned, the role of the teacher, and the use of Lesson Study as a means of professional development, with a view to widening its implementation in Australia.

What is Japanese Lesson Study?

Japanese Lesson Study is a voluntary professional learning activity whose origins can be traced back for almost a century. Lesson Study occurs across many curriculum areas, in the vast majority of elementary schools, and to a lesser extent in junior secondary schools and much more rarely in high schools.

Lewis (2002) describes the *Lesson Study Cycle* as having four phases:

1. goal-setting and planning – including the development of the Lesson Plan;
2. teaching the "research lesson" – enabling the lesson observation;
3. the post-lesson discussion; and
4. the resulting consolidation of learning, which, according to Lewis and Tsuchida (1998) has many far-reaching consequences

Lesson Study occurs in a variety of settings. Probably the most popular form of Lesson Study occurs within a single school, over a period of one or more years. Schools will decide on a goal and a curriculum area on which to focus. This goal-setting and planning phase begins with looking at broad goals, rather than fostering specific academic skills. For example, among sample goals given by Fernandez and Yoshida (2004) are the following: "Using a Japanese language class to foster students' ability to wrestle with topics they discover on their own" and "Developing well-thought-out mathematics lessons that provide students a feeling of satisfaction and enjoyment of mathematical activities, while fostering their ability to have good foresight and logical thinking" (p. 12).

Working in small groups over a year, teachers from different year levels might undertake three or four Lesson Study Cycles in which they plan a research lesson. One member of the group teaches the lesson, which is observed by teachers from the whole school, as well as possibly parents and outside observers, together with an outside adviser, who sometimes would have been involved to a minor extent in the early stages of the planning as well. Outside advisers might be university-based experts, regional instructional superintendents who specialize in the chosen curriculum area, or experienced teachers released for a year to provide staff development. However, in most primary schools there would also be an internal "expert" in the area – a teacher whose university major in their teaching degree was in that curriculum area.

Each such research lesson is followed by a post-lesson discussion, during which the teacher and all observers publicly reflect on the lesson and offer suggestions for how it could be improved. These reactions are based on detailed observations of the students' and the teacher's actions during the lesson. In some cases, the lesson might be revised and taught to another class, but this is not an essential part of the Lesson Study Cycle.

This pattern, which is typical of what happens in so-called "local" schools, is often extended in the more prestigious National Schools or schools attached to nearby universities. These schools may hold "open days" where teachers come from nearby schools or even from other cities across the country. Sometimes, as was experienced by one of the authors at a junior secondary school attached to a university, the open day involved research lessons being conducted simultaneously across a wide range of curriculum areas.

Other venues for Lesson Study observed include a Saturday Lesson Study "conference" at an open-plan primary school, where about 1000 participants observed and reflected on three sets of five parallel lessons in mathematics – a strenuous experience involving a lot of standing in extremely hot, hugely overcrowded conditions! In this case, the lessons were taught to children who were not known to the teachers. The teachers were either well-known "veteran" teachers, often experimenting with new ways to teach particular content and looking for suggestions from the observers, or teachers wanting to disseminate their own innovative ways of teaching to a wider audience.

In Japan, the process of Lesson Study is regarded as making participants and observers think quite profoundly about specific and general aspects of teaching. It is a long-term activity, not just about improving a single lesson, but rather about professional learning through participation in the whole process.

The Japanese Structured Problem-Solving Lesson

In mathematics, the research lesson, at least at the primary school level, usually follows the typical lesson pattern for a Japanese "structured problem-solving lesson".

According to Stigler and Hiebert (1999, pp.79-80), such lessons can be described as having the following stages:

- Reviewing the previous lesson
- Presenting the problems for the day
- Students working individually or in groups
- Discussing solution methods
- Highlighting and summarizing the main point.

Major characteristics of such lessons include:

- the hatsumon – the thought-provoking question or problem that students engage with and that is the key to students' mathematical development and mathematical connections;
- kikan-shido – sometimes referred to as the "purposeful scanning" that takes place while students are working individually or in groups, which allows teachers not only to monitor students' strategies but also to orchestrate their reports on their solutions in the neriage phase of the lesson;
- neriage – the "kneading" stage of a lesson that allows students to compare, polish and refine solutions through the teacher's orchestration and probing of student solutions; and
- matome — the summing up and careful review of students' discussion in order to guide them to higher levels of mathematical sophistication (see, for example, Shimizu, 1999).

A Research Lesson in Grade 3

While research lessons are usually planned to take 45 minutes, many such lessons continue for longer. The breakdown of time for the different stages of such a lesson often comes a great surprise for Western observers. For example, in one Grade 3 lesson observed, there was no review of the previous lesson, 45 minutes were spent discussing solution methods, and 5 minutes were spent on each of the other stages.

The problem presented was:

> *There are 35 pieces of cookies and 7 people. If each person gets the same number, how many pieces does each person get?*

How can such a prosaic problem lead to 45 minutes of discussion of solutions?

Firstly, students are accustomed to providing a wide range of solutions to problems and participating in extended discussions of their strategies. They understand that their solutions are listened to by the teacher and the other students and that they form an important vehicle for the learning that takes place in the class.

Secondly, there is much greater use of diagrams and drawings of solutions than is common in Western countries. A few examples of children's solutions are shown below (Figures 1 to 4).

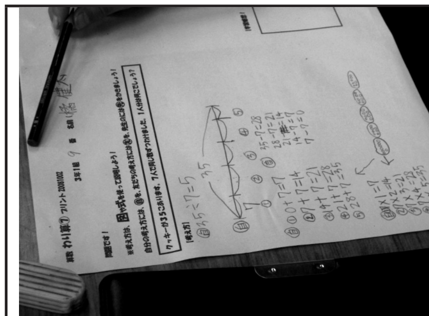

Figure 1. A solution for the cookies problem

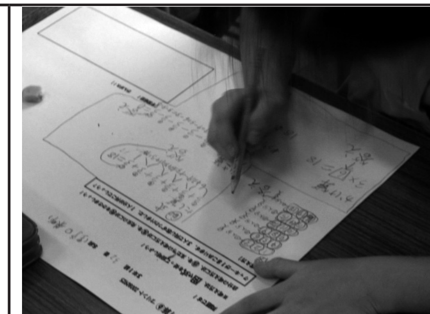

Figure 2. A different solution for the cookies problem

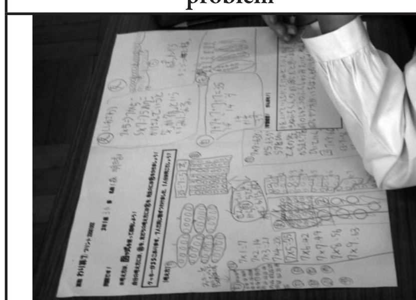

Figure 3. A child's solution, notes and summary of the lesson

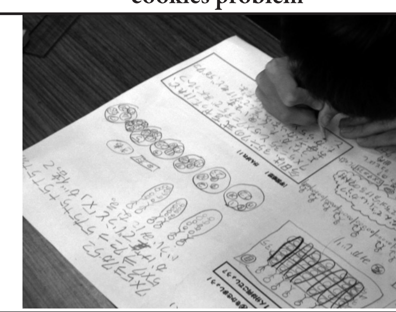

Figure 4. Another child's solution, notes and summary of the lesson

Another really important point is that the *hatsumon* – the thought-provoking question or problem – is not really how many cookies each person gets, but the different ways that a solution can be found for this problem. Children not only know the answer to 35 ÷ 7, but they understand that the lesson is leading to something much more important than finding this answer – in this case the lesson was an introduction to partition or "sharing" division with the aim of showing how multiplication can be related to such division problems.

While the children are working individually, the teacher is "purposefully scanning" their solutions – *kikan-shido* – in order to select children and decide the order in which they will come to the front to share their solutions.

As different children share solutions in the neriage stage of the lesson, the others make their own notes, as can be seen in the top right hand corners of children's sheets of paper in Figures 3 and 4.

At the end of the lesson, after the teacher has summed up the lesson and carefully commented on children's solutions – *the matome* — children write their own summary in the bottom right hand boxes on their sheets of paper (see Figures 3 and 4).

The Role of the Teacher

Doig, Groves and Fujii (submitted) identify four types of tasks typically used in Japanese Lesson Study research lessons – tasks that:

- directly address a concept;
- develop mathematical processes;
- have been chosen based on a rigorous examination of scope and sequence; and
- address known misconceptions.

In Japanese research lessons, the process of selecting the problem or task for the problem-solving activity comes about through *kyozaikenkyu*, which is the investigation of a large range of instructional materials, including textbooks, curriculum materials, lesson plans and reports from other lesson studies, as well as a study of students' prior understandings "which makes it possible for teachers to be able to anticipate students' reactions and solutions to the problems students study during lessons" (Research for Better Schools, n.d.). While all teachers need to engage in *kyozaikenkyu* as part of their lesson planning, Lesson Study requires teachers to engage in it in much more depth.

Watanabe, Takahashi and Yoshida (2008), remind us that the purpose of Lesson Study is not just to improve a single lesson, but to improve mathematical instruction in general, which involves careful attention to kyozaikenkyu, something that is not always attended to in non-Japanese Lesson Study. While the literal meaning of *kyozaikenkyu* is the study or investigation (kenkyu) of instructional materials (*kyozai*), the word *kyozai* means much more than textbooks or curriculum materials and needs to involve learning goals. According to Yokosuka (1990)

It is important that *kyozai* and subject matter content (specific knowledge and procedures to be learned through lessons) are distinguished. It is possible to explore the same subject matter with different *kyozai*, or we can investigate different subject matter with the same *kyozai*. (p. 19, translation cited in Watanabe, Takahashi and Yoshida, 2008)

Furthermore, according to Watanabe, Takahashi and Yoshida (2008), "*Kyozaikenkyu*, is the process to help teachers gain a deeper understanding of *kyozai*". It is

> the entire process of research activities related to kyozai, beginning with the selection/development, deepening the understanding of the true nature of a particular kyozai, planning a lesson with a particular kyozai that matches the current state of the students, culminating in the development of an instructional plan. (Yokosuka, 1990, p. 73, translation cited in Watanabe, Takahashi and Yoshida, 2008)

Thus it is very important that teachers have a knowledge of a range of tasks and the possibilities the tasks offer to meet their goals.

Lesson Study and Professional Learning

Lesson Study is much more than just planning together and observing one another's lessons. The post-lesson discussion, which typically takes about one hour and starts with reflections by the teacher who taught the lesson, followed by questions and comments from all observers, and ends with comments from the outside reactor(s), is a critical part of the process.

Isoda, Stephens, Ohara, and Miyakawa (2007, p. xvii) identify three key ideas underpinning Japanese Lesson Study:

- the idea that teachers can best learn from and improve their practice by seeing other teachers teach;

- the expectation that experts in pedagogy should be encouraged to share their knowledge and experience; and
- a focus on cultivating students' interest and on the quality of their learning.

While Lesson Study is not a compulsory feature in Japanese schools, it is clearly supported within schools and by the school system, with all teachers being expected to take part in professional learning and make efforts to improve their lessons. The fact that teachers must remain at school until 5 pm, even though students leave school much earlier, enables time to be set aside for Lesson Study activities. Moreover, it is possible in Japan to hold Lesson Study during the last period of the day in one class while sending the other children home, so that all teachers at the school are able to observe the lesson and take part in the post-lesson discussion. Publishers also play an important part by publishing teachers' lessons that have been developed through Lesson Study. In addition, journals such as the *Journal of Japan Society of Mathematical Education* regularly devote a section to Study on Teaching Materials. The clear involvement of university-based and other outsiders is also seen as not only a supporting mechanism for Lesson Study, but also as a way of breaking the isolation experienced by some school teachers.

How Lesson Study Could Work for You

Lesson Study requires a commitment of time and resources from groups of teachers from within a school or across schools interested in improving students' learning. While first hand experience of Lesson Study in the Japanese context or involvement of Japanese participants has often been found to be a key factor in the success of Lesson Study in the USA and elsewhere, there are now many online and other resources available to help potential participants understand the process. The involvement of an appropriate adviser is also critical to the success of Lesson Study. Arguably, however, one of the major resources needed is time release to observe research lessons and to take part in the post-lesson discussion.

If you or your school is interested in taking part in Lesson Study, please email us at susie.groves@deakin.edu.au or badoig@deakin.edu.au

References

APEC-HRD Lesson Study Project (n.d.) *Specialists on APEC Lesson Study: What is going on in each economy?* Retrieved 4 January 2010 from http://www.criced.tsukuba.ac.jp/math/apec/people/

Clarke, B. & Sanders, P. (2009). Tasks involving models, tools and representations: Making the mathematics explicit as we build tasks into lessons. *Australian Primary Mathematics Classroom*, 14(2), 10–14.

Department for Children, Schools and Families (2008). *Improving practice and progression through Lesson Study: Handbook for headteachers, leading teachers and subject leaders.* Nottingham: DCSF Publications. Retrieved 15 January 2010 from http://nationalstrategies.standards.dcsf.gov.uk/node/132730

Doig, B., Groves, S., & Fujii, T. (submitted). *The critical rôle of task development in Lesson Study.*

Fernandez, C., & Yoshida, M. (2004). *Lesson study: A case of a Japanese approach to improving instruction through school-based teacher development.* Mahwah, NJ: Lawrence Erlbaum.

Hollingsworth, H., & Oliver, D. (2005). Lesson study: A professional learning model that actually makes a difference. In J. Mousley, L. Bragg & C. Campbell (Eds.), *Mathematics – Celebrating Achievement. Proceedings of 2005 MAV conference.* (pp 168–175) Melbourne: MAV.

Isoda, M., Stephens, M., Ohara, Y., & Miyakawa, T. (Eds.) (2007). *Japanese lesson study in mathematics. Its impact, diversity and potential for educational improvements.* Hackensack, NJ: World Scientific.

Lewis, C., & Tsuchida, I. (1998). A lesson is like a swiftly flowing river: Research lessons and the improvement of Japanese education. *American Educator,* 14–17 & 50–52.

Lewis, C. (2002). *Lesson Study: A handbook of teacher-led instructional change.* Philadelphia, PA: Research for Better Schools.

Pierce, R. & Stacey, K. (2009). Lesson study with a twist: Researching lesson design by studying classroom implementation. In M. Tzekaki, M. Kaldrimidou, & C. Sakonidis (Eds.). *Proceedings of the 33rd Conference of the International Group for the Psychology of Mathematics Education* (Vol. IV, pp. 369-376). Thessaloniki, Greece: PME.

Research for Better Schools (n.d.). Glossary of Lesson Study terms. Retrieved 20 January 2009 from www.rbs.org/lesson_study/glossary.php#study.

Shimizu, Y. (1999). Aspects of mathematical teacher education in Japan: Focusing on the teachers' roles, *Journal of Mathematics Teacher Education,* 2, 107–116.

Stigler, J. W., & Hiebert, J. (1999). *The teaching gap: Best ideas from the world's teachers for improving education in the classroom.* New York: Summit Books.

Watanabe, T., Takahashi, A., & Yoshida, M. (2008). *Kyozaikenkyu*: A critical step for conducting effective lesson study and beyond. In F. Arbaugh & P. M. Taylor (Eds.), *Inquiry into Mathematics Teacher Education* (pp. 131–142). Association of Mathematics Teacher Educators (AMTE) Monograph Series, Volume 5.

White, A. L. & Lim, C. S. (2008). Lesson study in Asia Pacific classrooms: Local responses to a global movement [Electronic version]. ZDM - *The International Journal on Mathematics Education,* 40(6), 915–925.

Yokosuka, K. (1990). *Jugyokenkyu yougo daijiten*. (Dictionary of lesson study terms). Tokyo: Tokyo Shoseki.

Yoshida, M. (1999). *Lesson study: A case study of a Japanese approach to improving instruction through school-based teacher development.* Doctoral dissertation, University of Chicago.

STUDENTS' AND TEACHERS' USE OF ICT IN PRIMARY MATHEMATICS

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As part of a large research study, the authors explored the use of ICT in rural and urban Victorian primary schools. Some forty-five teachers and nearly seven hundred students were surveyed and a small number of them were interviewed. An important feature of this study was the investigation of students' use of ICT at home as well as at school for mathematics. This paper reports the findings of some aspects of the study, together with implications for teaching and learning.

Introduction

The imminent introduction of the Ultranet, "a state-of-the-art Web 2.0 system that reflects the modern classroom by breaking down the traditional walls" (Media release, 2010), to Victorian schools prompted our 2009 small-scale base-line study investigating the ways in which primary students and their teachers were already using Information and Communication Technology (ICT) in the learning and teaching of mathematics. A significant feature of this study was that students were asked about their use of ICT both at home and at school.

This paper reports on some of the results of this study and its implications for teaching.

Background

Research has shown that ICT has been used with varying success to scaffold learning in schools (see, for example, Muspratt & Freebody, 2007; Selwyn, Potter, & Cranmer, 2009; Smeets, 2005). Selwyn et al.'s (2009) study on children's engagement with ICT inside and outside of the school context showed that children's engagement with ICT was often perfunctory and unspectacular, especially within the school setting. This prompted them to suggest that schools develop meaningful dialogue with students about future forms of educational ICT use. On the other hand, Becta (2009) found a significant positive association between students' home use of ICT, for educational purposes, and improved attainment in national tests for mathematics and English.

Thus, in order to examine these different findings in a Victorian context, the focus of our 2009 study was primary school teachers' and students' use of ICT for mathematics learning and teaching, at home and at school.

The project

A total of 45 primary school teachers (34 urban and 11 rural) and 676 Year 3 to 6 students (488 urban and 188 rural) from six urban schools and seven regional schools in two school networks participated in this project.

Participation in the project involved teachers completing an on-line survey, with a small number of teachers also participating in a half-hour interview in which they were asked to elaborate on their survey responses. These interviews were audio-taped and transcribed.

Students were asked to complete a 20-minute written survey in their classroom, with questions read aloud to them. The survey questions included:

- How often did you use [different types of] ICT tools for mathematics, at school and at home, during the preceding week? (Types included computers, calculators, and specific software such as Excel.)
- What do you think about mathematics, and the use of ICT to learn mathematics?

A small number of selected students were invited to take part in half-hour interviews, to elaborate on their written responses. As with the teacher interviews, these interviews were audio-taped and transcribed.

Findings

This paper compares students' use of computers and the Internet, at home and at school, in urban and rural contexts. Other aspects of the project findings are reported elsewhere.

Computer and Internet use at home and at school

Figure 1 shows a comparison of the responses relating to computer use for mathematics at home and at school during the week preceding the survey.

While it is not surprising that a greater percentage of students used the computer at school than at home, it is revealing that quite a large number of students (50%) say they used computer software for mathematics at least once at home in the preceding week.

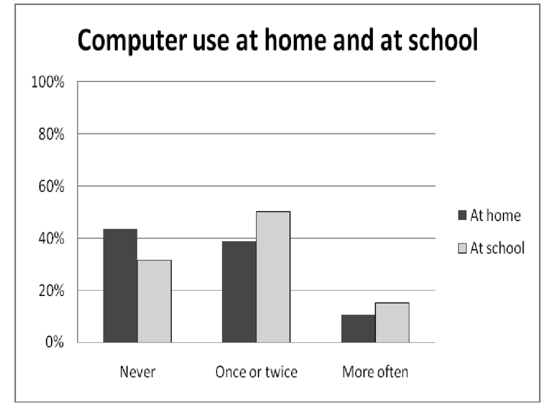


Figure 1. Frequency of computer use for mathematics at home and at school during the preceding week

Among those who indicated they had used computer software at school and at home, 79% said they used games at school to learn mathematics, while a slightly larger number (82%) used games at home (see Figure 2). *Mathletics,* a web-based mathematics educational site, had been used at home by about 30 % of students. Although this proportion may not seem large, it is interesting to note the number of students who were engaged in doing mathematics at home on-line.

A smaller percentage (19%) of students also used tutor programs at home for mathematics. One student said "Dad bought a $6 000 tutoring program .... It is called the *Mathemagic Computer Tutor.* It helps you with algebra, percentages. Then we have English, it helps you with spelling, vocabulary". Other software mentioned were *SmartKiddies* and *Maths Circus.*

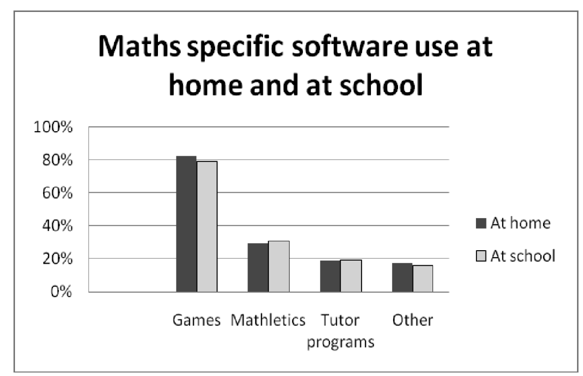


Figure 2. Maths specific software use at home and at school

Of, perhaps, more interest is the proportion of both Mathletics and tutoring programs being bought and used by schools (31% and 19% respectively). It raises the question, for teachers, of how consistent children's mathematical advice really is, and who is the main source of their learning: the home tutor, or the classroom teacher!

Generic software was also used for mathematics at school and at home (see Figure 3), the most common being *Word* (40%) and *Excel* (32%). A Year 3/4 boy said "Last term we used *Excel* to find out what country most people originated from in our grade … We printed out all sheets. We had to write down what country most people came from in our books … We made the graph on Excel … Like, what percent of people." A Year 6 girl said "We sometimes for maths, use *Word* where she [the teacher] makes objects which have fractions and decimals".

At home, the frequency of use of these software packages was generally slightly lower than that at school (*Word* 37%, *Excel* 27%). A small percentage used *Microworlds* and other generic software in mathematics such as *PowerPoint.* One Year 5 boy said "We use *Word* at home for writing stories and *PowerPoint* at home for making slide-shows about myself telling everything about me, what I like doing … [for mathematics]. In *Word,* I make a chart as well as *Excel".*

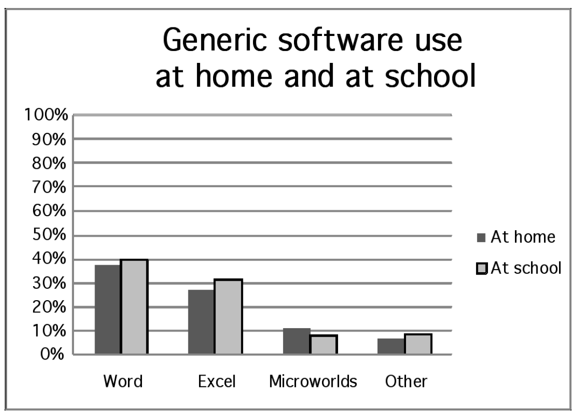


Figure 3. Generic software used for mathematics at home and at school

Most students used the Internet for mathematics at least once in the preceding week, 50% using it at home and 70% at school. These uses included searching for information, using electronic mail (e-mail), using mathematics sites (e.g. *A Maths Dictionary for Kids, Coolmaths for Kids*) and blogs. Other uses of the Internet included playing mathematics games (e.g. fraction games). One Year 3/4 boy said "*Maths 300* that we have on school computer. It has fun maths games. There is one, *Funbrains* … There are little maths questions: division, plus, take away … [I use it] once a week at home …. At school, we have computer lab on Wednesday. We do it for half hour every week".

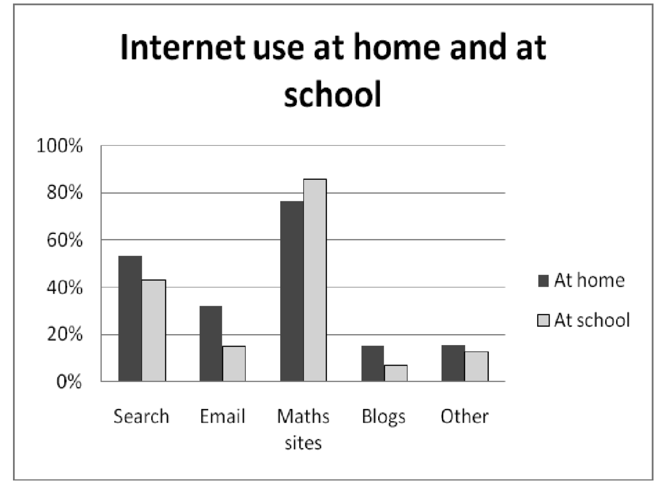


Figure 4. Internet use for mathematics at home and at school

More students used the Internet for searching, e-mail and blogs at home than at school (see Figure 4). This may not be surprising given that students have more time at home to do searches, send e-mails and write blogs. It is also not surprising that more students used mathematics sites at school than at home, but what was surprising was that 76% of students used mathematics sites at home and that they used the Internet in such a variety of ways at home.

One Year 5/6 student said he went to the Internet "a couple of times in a week … to play games … Sometimes I get a chart from the Internet, like Roman Numerals that will help me in homework in future like in High School". Although the numbers are small, students also used blogs. One Year 6 girl said "Now we do [use blogs]. We have just started using *Glogster* in the last couple of days".

Computer and Internet use in rural and urban regions

Urban students' use of computer software did not differ significantly from those of students in rural schools with 66% of urban students saying they used computer software for mathematics at least once in the preceding week compared to 64% of rural students (see Figure 5).

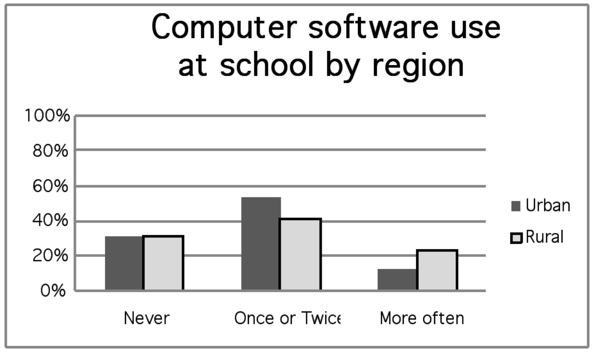


Figure 5. Computer software use for mathematics at school by region

The use of computer software at home differed only slightly between urban and rural students, with 47% of urban students saying they used computer software for mathematics at least once during the preceding week, compared to 55% of rural students. This difference, however, is not significant, particularly given that a higher percentage of urban students never used computer software for mathematics at home.

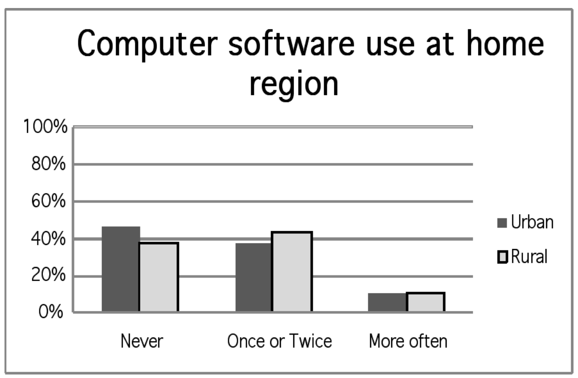
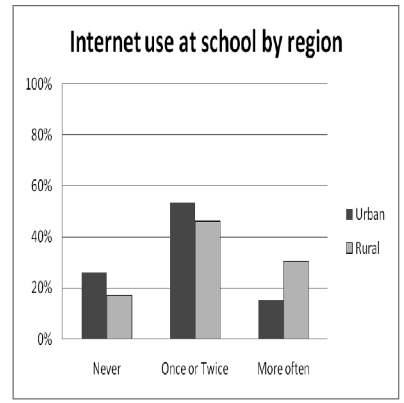


Figure 6. Computer software use at home by region

However, the situation was different with Internet use. Chi square test showed that there was a significant difference in Internet use by students in rural and urban schools (χ^2= 20.462, df = 2, p= .00) – see Figures 7 and 8. A comparison of responses from students in urban and rural schools shows that 77% of students in rural schools used the Internet for mathematics at school at least once during the preceding week, compared with only 68% of students in urban schools.

On the other hand Internet use at home was not significantly different for students in the two locations, with 44% of rural students saying they used the Internet at home at least once compared with 53% of urban students. This seems to suggest that where there is access to the Internet in rural locations, students are in no way less engaged with the Internet at home than their urban counterparts. What was surprising was that the use of the Internet at school was significantly higher for rural students compared to urban students. This could, perhaps, indicate that rural students and their teachers are more inclined to use the Internet in the teaching and learning of mathematics given easiness of access to information via the Internet.



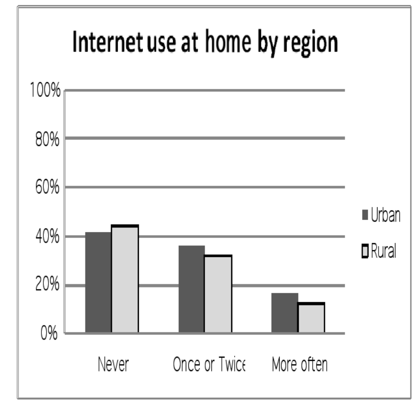


Figure 7. Internet use at school by region *Figure 8. Internet use at home by region*

Conclusion

Similar to Selwyn et al.'s (2009) study, we did not find computer and Internet use in school settings to be spectacular, but were impressed by the fact that half of the students surveyed use computer software and the Internet at home for mathematics. This has implications for teachers. Where students have access to computers and the Internet, they are often motivated enough to use tutoring programs, software, games and the Internet for mathematics at home. From the interviews, it seemed that family affluence and parental guidance plays a major role in promoting the use of ICT at home. Teachers could work in tandem with parents to promote the use of ICT for mathematics at home, given that computer games seem to be a major draw-card for students. *SmartKiddies* seems to be a popular website among the students surveyed, but there was no mention of the The Learning Federation by students, although most teachers in Victoria have access to this website.

It will be interesting, too, to follow the development of ICT, as Professional Learning for teachers becomes more available, and the amount of technology in the classroom increases. Or, should we wait for some of the current generation of primary school children to become primary teachers?

References

Becta (2009). *Enabling next generation learning: Enhancing learning through technology.* Retrieved10 August 2010, from http://publications.becta.org.uk/display.cfm?resID=39140&page=1835

Media release from The Premier of Victoria (2010). *Pike flicks the switch on Ultranet revolution.* Retrieved 23 May 2010 from http://www.premier.vic.gov.au/component/content/article/10331.html

Muspratt, S. & Freebody, P. (2007). *Students' perceptions of the characteristics of "good" and "poor" digital learning objects.* Paper presented at the AARE Annual Conference: Research Impacts: Proving or Improving?

Selwyn, N., Potter, J., & Cranmer, S. (2009). Primary pupils' use of information and communication technologies at school and home. *British Journal of Educational Technology,* 40(5), 919-932.

Smeets, E. (2005). Does ICT contribute to powerful learning environments in primary education? *Computers and Education, 44,* 343-355.

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DIDACTIC BY STEALTH

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When you discover a mathematical idea that could revolutionise an aspect of the game you love (Australian Rules Football), you just want to share what you have learnt. When a theoretical approach draws blank stares from football practitioners, the search for a strategy to find a palatable presentation of your ideas must begin. Suitable practical activities and modelling exercises and in the right order with a minimum of numbers.

A Chance Discovery

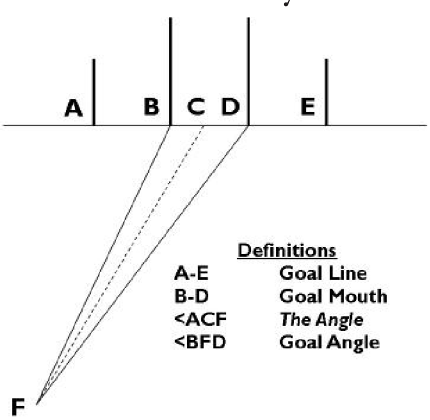


Figure 1

In 1997 I was given a Year 7 maths class to teach. As a long time Chemistry, Biology and Science teacher, this was a refreshing challenge. Working in a small school, I had already turned my hand to Theatre Studies and Woodwork so I thought: 'What the heck?'. I hunted about for a few real life applications of angle theory.

I decided to check out the angle faced by footballers as they lined up for goal. The angle I felt most pertinent was that defined by the two posts and the point of kicking (Figure 1). I dubbed it, the 'Goal Angle'(<BFD). This is not to be confused with *The Angle (<ACF)* referred to by football commentators. ("He's on a slight angle" meaning close to perpendicular to the goal line (the line joining the goal posts). Or "He's on a difficult angle" meaning say at 20 degrees to the goal line.

I randomly selected quite a few kicking positions and drew a few lines to the goal posts, measured the angles and soon discovered that a pattern emerged. I was seriously intrigued. All the points on the ground where the Goal Angle was, say, twelve degrees formed a circle that (at least theoretically) included the posts themselves (Figure 2). Similarly, a set of points for eight degrees of Goal Angle also formed a circle – a larger one. I was giddy with excitement. Truly. Each Goal Angle proscribed its own circle. In my ignorance of any snappy pre-existing name, I dubbed these *anglecircles*. It was a Saturday and a maths colleague was at work. I just had to tell her! She looked at my discovery and declared, quite nonchalantly, that it was an application of a standard Year 10 principle where if you popped a chord across a circle, lines drawn to the ends of the chord from anywhere on the circle created (or, indeed, subtended) the same angle.

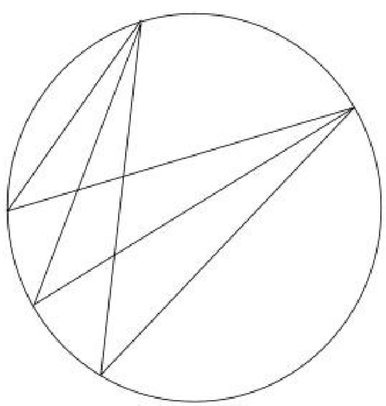


Figure 2

I thought the ramifications of this discovery were huge for Australian Rules football. The received wisdom was and, I think, still is that the 'greater' *The Angle,* the greater the difficulty a player would have kicking the ball between the posts. (The ambiguity of football commentary language could render a mathematician aghast – that is. a slight angle versus a tight angle.) This was essentially regardless of the distance the player had to kick the ball. I felt that the demand for accuracy was defined by my Goal Angle, not *The Angle,* as convention would have it.

Sharing a Discovery

During the next few weeks, I proselytised my idea to anyone standing still. (I also secretly waited for someone to tell me that they already knew this. It never happened.) I explained the implications for the set-shot in Australian Rules football. For example, a footballer 'straight-in-front' at 46 metres has a Goal Angle of 8 degrees. (Commentators would say 'he has no angle to contend with'). I then went on to explain that a player 'on a forty-five degree angle' 32 metres out also has a Goal Angle of 8 degrees as well. (Commentary – 'but he'll have some angle to deal with'). Contrary to conventional wisdom, the second option should be easier than the first. Maths teachers tended to 'get it' straight away. They trusted maths. The majority glazed over. At best they might concede 'You are right in theory but it is different on the field'. Occasionally they would resort to offering the 'left bower' by asking "So did you play the game?". I shook my head and theirs went all smug. Clearly my missionary zeal was not enough.

Climbing over The Brick Wall

What was the point of a theory if those most likely to benefit from it would simply reject it out of hand? It is fair to say that most footballers don't put mathematical curiosities high on their list. Could I be didactic by stealth?

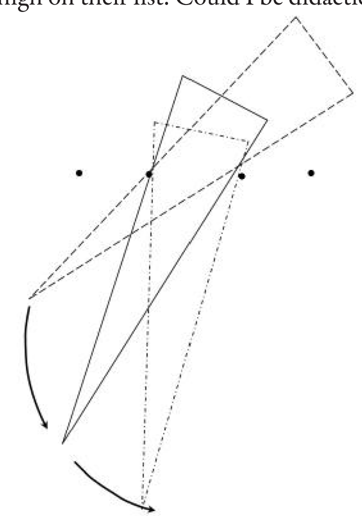
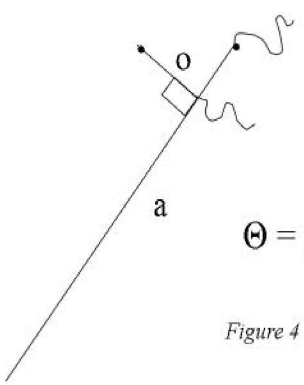


Figure 3

I decided to make a 100:1 model of a ground complete with goalposts 6.4 mm apart. I fashioned wedges of various angles with an on-board pencil which I poked through the goals onto the field area. By carefully dragging the wedge through the goals, a perfect circle would be drawn. Clearly, from any point on this *'anglecircle',* the Goal Angle must be the same (Figure 3). Narrow wedges produced big circles, fat wedges produced small circles. Shown to most 'regular joes' this was interesting but not enough to challenge the stubborn conventional thinking. Perhaps it was time to venture out onto the actual football field.

Making circles on the grass

A giant wedge would be problematic. Drawing circles, however, was easy. A peg in the ground about 20 metres straight in front, a bit of rope, walk to a goalpost, tighten the rope and walk a circle dropping witches' hats as you go around to the other goalpost. Now to measure the actual Goal Angle. The simplest method was to run tape measure from a selected kicking point on the circle to the inside of the far goal post. Then another tape is run from the near goal post to the first tape meeting it at right angles. This is actually easier than it sounds. You don't need anything other than your eyes to find the right angle. Hey presto – right angled triangle. Now a bit of Year 10 trigonometry. From the intersection, measure the distance to the kicking point and to the near goal post. Inverse tan ratio (Figure 4). There's the angle. Measured *on the paddock!* Try a few other positions on the circle and the theory is confirmed This works for a big circle or for small circles. Stubborn conventional thinking despatched to the boundary? Not as yet. Interesting for students "and we get to go outside", but that is all.



$$\Theta = \tan^{-1} o/a$$

Figure 4

So would a suitably enlightened footballer paid a mark 30 metres out on a bit of an angle whip out the tape measures and calculator and crunch the numbers? No. Not on your Nellie. Perhaps the commentator, equipped with a suitable map of the ground with *'anglecircles'* might estimate the goal angle and inform the listener as to the level of accuracy required to 'put it between the big sticks'. "He's thirty five out on a bit of an angle, Butch. What's his goal angle from there?" Butch responds "He can see 12 *degrees of daylight* from there Drew. He'll only need to be pretty straight."

A handy protractor

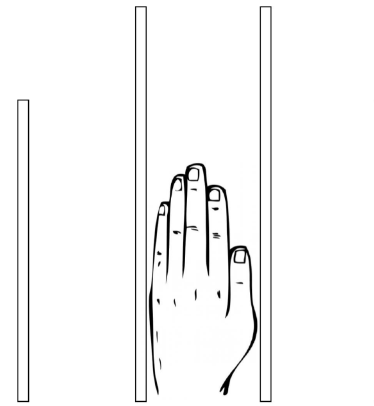


Figure 5

So how can the footballer measure the goal angle? Maybe an answer might come from the stars, or at least astronomers. In the field at night astronomers have for ages described positions of heavenly bodies by extending their arms in front of them and quoting a star as being a certain number of fingers above the horizon. Can a footballer use their fingers to measure the angle between the goals? Why not? Simply raise the hand not holding the ball and assess the *digits of daylight* between the goals? (Figure 5) For me, each finger is very close to two degrees of angle. That means four fingers means eight degrees. That is of little importance to the footballer however. They may as well adopt 'the digit' as the unit. (We only have 360 degrees in a circle because it is close to the number of days in a year.)

But to be frank, why would they? Mathematical curiosities don't rank high on most footballers priority. Not unless it can help to win *The Flag*. Recently many have noted that the set-shot is the one aspect of Australian Rules that has not changed in 100 years. They also argue players have barely improved its execution in that time. Perhaps, properly packaged, a mathematical analysis may provide a psychological edge.

Players already know how far they must kick. Perhaps it would help if they also knew how accurate (quantitatively) they must be.

A Change of Approach

Step One -Target Practice

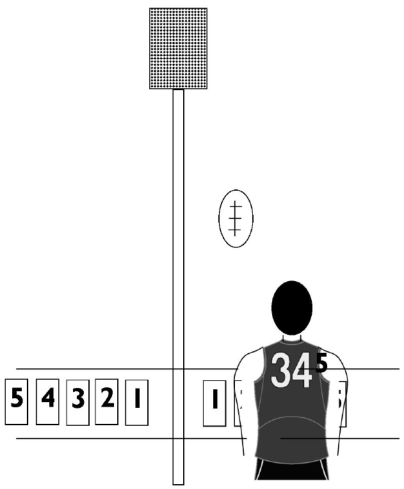


Figure 6

So here goes. Firstly, how accurately can you kick big fellah? "Oh, ah, I'm pretty good." Enter a mathematical analysis. The footballer kicks at a single target (a light pole that most grounds have) from, perhaps, 40 metres. (Figure 6) The deviation from the target is measured in *digits of daylight* (DODs). Markers are placed on the boundary fence (1.4m apart) and a cumulative frequency performance table is filled in. The footballer is supplied with a profile of their accuracy. They then kick from a different distance, maybe 30 metres. The markers will need to be shifted. (1.05m apart) Deviation from the target is measured in units of *angle* not distance. This is *critical* and probably novel.

The footballer is now equipped with knowledge of their accuracy from a *variety* of distances. For example "From 40 metres, 30% of the time I'm within 1 DOD of the target, 60% within 2 DODs, 90% within 3 DODs and 100% within 4DODs." (Figure 7 – table below.) In a training session, players might compete with each other. Perhaps a lollypop as a prize. (Cold and hops flavoured for the older ones.) At this point descriptors can be employed. Being within one DOD means the kick is 'very accurate'. Two DODs, simply *'accurate'* , three DODs *is 'pretty straight',* four DODs gets called *'pretty average'* and if you are just inside the 5 DOD marker, well the kick must have been *'dodgy'.* This makes the maths accessible to Joe Average. The employment of descriptors selected from current football parlance aims to reduce the barrier between maths and the average football lover, be they player or spectator.

Target Practice *(Figure 7)*

Analysis of accuracy in projectile launching by finger width

Projectile – Football
Date – July 27 Talent – Boris Bingwharr Distance – 40 metres

Finger width tolerance	Trial #1	Trial #2	Trial #3	Trial #4	Totals	Category %	Cumulative %
1 finger	xxxx	xxx	xxxx	xx	13	32.5	32.5
2 fingers	xxx	x	xx	xxxxx	11	27.5	60
3 fingers	xx	xxx	xx	xx	9	22.5	82.5
4 fingers		xx	xx	x	5	12.5	95
5 fingers	x	x			2	5	100

Projectile – Football
Date – July 27 Talent – Boris Bingwharr Distance – 30 metres

Finger width tolerance	Trial #1	Trial #2	Trial #3	Trial #4	Totals	Category %	Cumulative %
1 finger	xxxxx	xxx	xxxx	xxxxx	17	42.5	42.5
2 fingers	xxx	xxxxx	xxx	xxx	14	35	77.5
3 fingers	x	xx	xxx	x	7	17.5	95
4 fingers	x				1	2.5	97.5
5 fingers				x	1	2.5	100

Step Two -Wedge kicking.

Footballers are invited to measure the angle and then kick the ball into a wedge defined by rope lying on the ground. Wedges of 2, 4, 16, 8 or even 10 DODs are tried. A central target is placed some distance out. I used a 1 square metre cartoon target called 'Eileen'. She is knitting and I tell the kickers to aim at her as she sits 'in the stand'. This is the conventional wisdom from countless successful goal-kickers such as Peter Hudson. When players kick the ball, they either do or don't keep it in the wedge. Pass or fail. There is *no* sense of kicking at goal. Just aiming for a target and noting the success. They can also compare how they are going compared with the Target Practice work earlier by consulting their personal performance data. They should use this data to predict their likely success.

Other wedges are arranged. A way to quickly change the angle is critical. Footballers have limited patience. Note that at this stage, *goalposts* have not got a look in. The exercises so far are to make the footballer aware of how accurately they kick. That is all. (Don't mention the "Goal" word.)

Step Three – Introducing goalposts by stealth.

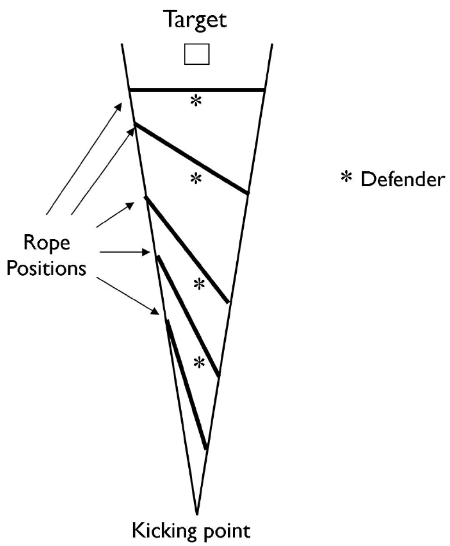


Figure 8

Now settle on an angle of 4 DODs or 8 degrees. A 'man on the mark' is added. A 'defender' is added at various but fixed distances imposing on the kicker a minimum distance since the kick must pass over their outstretched arms. Another rope that measures 6.4 metres length is also introduced immediately behind the defender. (Figure 8) It is placed in a range of positions stretched out between the original ropes. Clearly it will need to be on some oblique angle to reach between them. The footballer hopefully thinks you are weird and keeps on lobbing the ball past the defender straight over 'Eileen'. Ideally, they consider this third rope irrelevant. They are always encouraged to measure the goal angle and how the rope, obviously, does not affect it. I mean, duh! It is vital at this stage to establish that the closer the rope is to the kicking point, the easier it is to make the ball clear it. Perhaps some footballers may now twig. You have replicated the various positions on a *anglecircle* but simple given them a common kicking point. Careful discussion of the relative difficulty of the tasks is very important. The footballer should be encouraged to acknowledge that accuracy required remains unchanged. They may agree that with the defender closer, the task is in fact easier.

Goal posts are introduced. I use P.A. speaker stands with some PVC pipe attached. They are placed one on each rope in positions 6.4 metres from each other. (I marked the ropes with markers for this purpose selected to coincide with the traditional 15, 30, 45, 60 degree versions of *The Angle*) This point is critical. The footballer should now ignore the goalposts just as they happily ignored the third rope. This will not be easy. All the best goal kickers including the aforementioned Peter Hudson advise ignoring goalposts however. It

is not hard to believe that psychologically, kicking between two posts is vastly different to hitting a single target. In fact aiming at a single target is what footballers do in general play. And they do this with incredible accuracy.

What has been done now is to effectively place a series of kicking points from an *anglecircle* on one wedge.

Having cunningly got the footballers to 'admit' that the position of the rope makes no difference, it is time to coax them to believe that the goalposts position should equally make no difference. All this assumes they are kicking a straight kick from the kicking point. I like to suggest that 'the final siren has gone, two points behind, no playing on". It helps keep the maths simple.

Step Four – Getting Real

It is time to venture to the forward line where the real goal posts are. An interesting exercise could be to have a dozen players head off walking backwards from the goalsquare radiating in all directions and stop when they find a point where the goal angle is 6 DODs. Lo and behold, they are standing in a circle. (Figure 9) (I have manufactured some standardised arm and finger substitutes to make this work for a mixture of body proportions. I also have a calibration board so as to allow anyone to standardise their own hand and arm.) Some are *'on a severe angle'* while others have *'only a slight angle'*. I ask them to think about the relative difficulty of their tasks. They all have 6 DODs or a goal angle of 12 degrees. It's just that some *must* kick the ball further. The longest kick is from *straight in front* with, supposedly, 'no angle to deal with'.

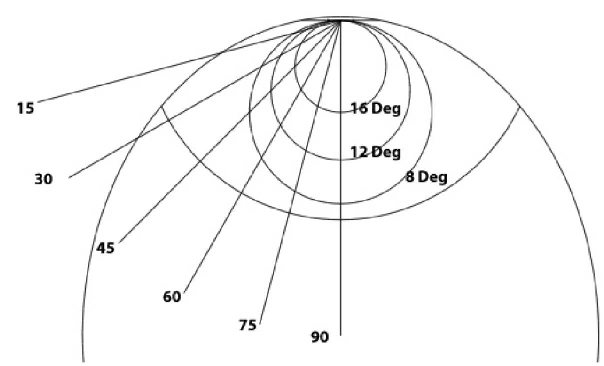


Figure 9

It is worth noting that in Australian Rules, there is no maximum distance for a kick. By now, they will also, hopefully, note that as they walk around the circle *they* created, they are essentially seeing the wedge with the goalposts in the various positions only spread around fixed goals. Or maybe they won't. These transpositions of the wedge are a tad abstract. This may need work. And patience. There is a lot of challenging of preconceptions at this point. Giving up long-held ideas requires guts or maybe a leap of faith.

The use of video can help. I have filmed the goalmouth from the back of a ute driving around an actual . The width of the goalmouth on the screen does not change although the goalposts' own appearance does vary.

I have also constructed an aid to mimic the ute doing the circle work. It is a 1:100 model of a forward line with a Lazy Susan that touches the goalposts (pop rivets with the 2mm shaft upward pointing). A video camera mounted on the rim of the Lazy Susan that is rigged to point straight at the goals. (I used a metal rod attached to the camera that pierces the pivoting but rotating goal umpire.) Hook up a monitor and rotate the Lazy Susan. The distance on the screen and hence angle between the goalposts does not move. This has proved to be the most convincing tool of all. Cheap and cheerful. The looks on the faces of viewers is priceless. The sound of pennies dropping is deafening.

Players are free to wander around the forward line and select a spot from which to evaluate their chances. Equipped with an accuracy profile that is based on goal angle and distance to the goals, a truly *scientific* assessment could be made. Even in the heat of battle, that is still possible. A quick DOD measurement will tell the player what accuracy is required.

As for media commentators, they seem to adopt new terms readily. Expressions like 'structures', 'the hot spot', zone defence were unheard of 20 years ago. The snappy nomenclature I have adopted may not suit media professionals. What would be tricky is to get 'expert commentators' to admit they have been wrong all these years. This theory shows that the goal angle from 50 metres (about 7 degrees) is demanding. Yet a miss from there attracts scorn from the experts in the commentary booth whereas a miss from close in but on 'a severe angle' is readily forgiven in ignorance of the *goal angle* of perhaps 10 or 12 degrees. Very do-able.

The role of maths is central in all this. The means of presenting the maths is crucial however. Maths often gets a bad rap for being esoteric. My aim has been to find a way to coax anyone at all who will listen towards mathematical enlightenment. Actually, to be truthful, I just don't want to see my team miss easy goals. We might win *The Flag* one day. I'd like to think I did my bit to help.

PROBLEM SOLVING – A GRADE SIX PRIMARY SCHOOL EXPERIENCE YEARS: 5 TO 8

Ian Bull

St Kevin's College

Problem solving in the mathematics classroom can present students with the opportunity of using different thinking processes as well as being lots of fun. It can be used to teach students a range of skills that are difficult to show in ordinary mathematics lessons. The problem solving program that has been run at St Kevin's College, as well as involving all students in the class, has been able to challenge all students and extend the high achieving students.

Thinking in the Mathematics Curriculum

Edward de Bono contends, "I do not suppose that there is an education system anywhere in the world which does not claim that one of the prime purposes of education is to "teach students how to think"" (delete on set of quotation marks). 1 Look inside some mathematics classrooms and you will probably find the majority of time is spent teaching facts and acquisition of basic knowledge exclusively. Where does creative thinking exist in this forum? Edward de Bono contends that "Education is essentially about the past. It is a matter of sorting, reviewing, describing and absorbing existing knowledge". 2 He goes on to say that "In a stable world it was enough to teach "information" because that would last for a student's lifetime. Socrates, and the other members of the Gang of Three (Plato and Aristotle), established the notion that "knowledge" was enough and that once there was knowledge then all else would follow".3 Knowledge and skills certainly need to be taught in the mathematics classroom, of that there is no doubt; however if that is all that is achieved then that is not enough. Clearly, possessing knowledge on its own is only the first step –

it is what can be achieved by applying that knowledge through appropriate activities that provides a reason for that study as well as fostering an enjoyment of the challenge.

A major problem in a knowledge-centred classroom is that students don't achieve a real level of understanding of what they are studying; concepts are not placed in a practical context. Often algorithms are rote learnt without having any practical basis for their existence. In these classrooms, assessment is centred about the repetition of these routines without an appreciation of their application to the real world. It is, of course, important that students are able to carry out basic procedures – arithmetic algorithms, multiplication tables facts, knowledge of geometric facts etc, but if this is where work stops in the mathematics classroom then this study is pointless, stunted and meaningless.

I propose that lateral thinking can be used to underpin and be used to pursue a deeper level of understanding of what is being studied in the mathematics classroom. Lateral thinking involves risk taking – by volunteering ideas that could be judged as being wrong but as Edward de Bono states "it is better to have enough ideas for some of them to be wrong than to always be right by having no ideas at all". 4 Edward de Bono also defines lateral thinking as "the sort of thinking that is concerned with changing perceptions and concepts". 5 In this way lateral thinking is about breaking out of established forms of thought to gain another view of what is being studied, thus increasing the perception of the concepts seen from different viewpoints which leads to a heightening of interest and fosters a deeper understanding of the core ideas. In this way it is crucial that we as teachers present our students with opportunities to think about what is being studied in different ways and to record the thinking processes that were used to perceive and solve the problems with which they were presented.

The message from Edward de Bono is clear; teaching students different ways to think is crucial to helping them take control of their lives and the discipline of mathematics can provide a perfect opportunity and vehicle to empower them to do so.

Student's thinking skills can be improved by

- giving them practice (**change to 'se' not 'ce'**) by encountering situations to be investigated and trying to express opinions as to their possible solutions, and
- working with and sharing ideas with other like-minded students, and
- having students record the ways that they thought about the problem, and
- getting students to think about where the investigation could be taken.

The challenge is to find the opportunities that can be used to bring these thinking skills to life.

Work at St Kevin's College Primary School

During the second semester of this year I worked with the middle third of students from the year six classes at St Kevin's College and presented them with a variety of problem solving tasks, designed to encourage the understanding, and application of their Thinking Skills in mathematics.

Students were presented with a number of tasks each centred on different topics. The first group of tasks, included later in this paper involved ratio and proportion where quantities were allocated to different people or groups according to a rule.

The group started with a WHOLE CLASS ACTIVITY, followed with a LESSON TASK, then an APPLICATION TASK and finally two EXTENSION TASKS. This approach is designed to guide students through the investigative process, acquainting them with the context of the task and helping them to go deeper into more complex situations. If they had been given the extension tasks at the start of the investigation, most students would not have known was expected of them.

In the preliminary CLASS ACTIVITY, students worked in pairs. Students were given a picture of some cubes in four piles. They were asked to write clues that the other person could use to draw the correct number of cubes in each numbered pile. We spoke about what clues would be allowed. For instance a clue could not be "there are three cubes in the first pile". The clues had to give the other person an idea that they could build on to eventually find the solution. Following this introduction each pair was given some pictures of different numbers of cubes in piles in an envelope. Each student hid their picture from the other student and wrote clues to describe number of cubes that were shown in the different piles. They exchanged the clues and each student had the challenge of drawing the correct number of cubes in the correctly numbered piles. Each student judged the effectiveness of the clues and students could amend and/or add to their clues in the process. The aim of this initial activity was to acquaint students with the idea that cubes could be allocated into different, specific piles as well as highlighting the use of communication skills to write clues to describe the situation rather than just describe what they saw. The students improved the sophistication of their clues – it really made them think about describing cubes in piles in a different way. It was the game/play element that made this activity work so well – they all had fun doing it and there was a real sense of challenge in being able to construct simple and (add comma and delete and) sensible but meaningful statements. Students also supported each other when the clues needed improvement, making suggestions as to how the clues could be improved. Students were praised for their ability to use the least number of clues in the most effective way.

The CLASS ACTIVITY set the scene for the next more formal LESSON TASK. This comprised of three questions, one with dividing money between two people and two more tasks where cubes were placed into piles according to clues. Students were then asked to solve these problems and write down what method they used – how they went about thinking about solving the questions. At this stage all students knew exactly what was happening, based on their experience from the previous activity and could concentrate on solving the problems as well as describing how they found their answer.

An APPLICATION TASK followed which was designed to apply the skills of the first two tasks where students were required to proceed in a more independent manner. The idea in question four where Henry having 25 cubes, puts some of them into four piles immediately(add comma) raised the issue of multiple solutions. The students were very quick to notice this, and I highlighted this point to the group after some students raised it with me. Completion of this task was set for homework and the solutions were shown on the interactive whiteboard the following week.

As part of my Problem Solving resource I wrote two EXTENSION TASKS which built on the ideas of the previous tasks and asked students to take those ideas to a higher more complex place. As always in my classes, I highlighted the need for students to express the thinking that they used to find solutions.

Results; a profile on the thinking demonstrated by the students

Students became enthusiastically involved with the idea of moving objects around according to sets of instructions. The skills required to complete the activity were not "taught" in the traditional sense and students were expected to find their own way to approach and solve the problems. Students were asked to record their thinking processes that they used to find their solution. Equal weighting was given to the thinking process used to solve the question as was the ultimate solution itself.

For instance, question 1 from Extension Task 1 revealed a rich learning experience for the group in terms of the ways that students derived their answers.

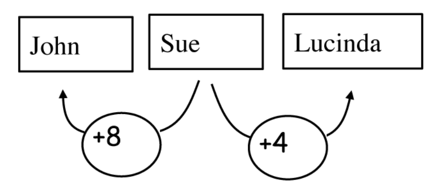
Question 1

John, Sue and Lucinda share a prize of $48. If John gets $8 more than Sue and Lucinda gets $4 more than Sue, find how much money each person will get.

The problem was not difficult and was quickly solved by all students. It was the thinking responses given by different students that I found to be exciting and illuminating. It is responses such as the ones that follow that have broadened my teaching experience over my career.

Solution 1

Some students drew the diagram to show the structure of the task and then used trial 'n (**change to and**) error to find the solution.



They then used trial 'n (**change to and**) error guessing the amount that Sue will get to find that Sue = $12, John $20 and Lucinda $16.

Solution 2

One year 6 student expressed the problem in algebraic form:

> Letting J stand for John, S for Sue and L for Lucinda: J = S + 8, L = S + 4, J + S + L = 48.

He then applied trial '(**change to and**) error as before (**delete before and change to demonstrated in solution 1**).

Solution 3

One student said "subtract $8 and $4 from $48 to give $36, then divide this ($36) by 3 as there are three people. This gives the average amount that each person will get – in this case $12. So Sue will get $12, John will get $12 + $8 = $20, and Lucinda will get $12 + $4 = $16.

I found the last student's response very interesting and pursued it further. I proved his technique in general terms as: Let Sue's amount be $x. Let John get $a more than Sue so that he gets (x+a) and let Lucinda get $b more than Sue so that she gets (x+b).

The total amount received by the friends will be x + (x+a) + (x+b) = 3x + a + b.

The student then said that this total, which happens to be $48 then needs to have a and b, or a + b subtracted = (delete) 3x + a + b – (a + b) = 3x, which is then divided by 3 to give 3x ÷ 3 = x which is the amount that Sue will get – (change to ; to avoid confusion about subtraction symbol) in this case $12. Adding a and b for John and Lucinda gives $12 + $8 = $20 for John and $12 + $4 = $16 for Lucinda as was found before. This proves that the method used by this student works for all cases. I was reassured by this student's solution process. If this student had been in year 9 or above, rather than year 6, I would have asked him to prove this contention.

Conclusion: results of trialling in primary and secondary schools

The materials and approaches outlined in this paper have been extensively trialed in a variety of schools, both primary and secondary in a number of states across Australia as well as at St Kevin's College this year both with whole classes and smaller withdrawal groups. It has been the first time that I have been able to use these materials first hand with my own students at (**add a**) primary school level as well as demonstrating their use to other grade six teachers at St Kevin's College.

In previous years the materials described in this paper were trialed in a range of both primary schools and secondary schools predominately in Victoria but also in Western Australia, New South Wales and Queensland. (**Not sure if you need this statement given the one in the paragraph above**) The findings reported by all parties were extremely positive where students responded in a very enthusiastic way to the tasks. This was seen as providing all students with appropriate challenge but more importantly providing the higher achieving group of students in the class with an avenue to apply their higher order thinking skills. The classroom lead problem solving section was cited as being pivotal in the process of introducing the idea of problem solving to students, allowing the use of further open-ended tasks in a less teacher focused manner.

References

Bull, I. (2010) - *Maths Problem Solving for Higher Achieving Students – Upper Primary / Lower Secondary*. South Melbourne, Vic: Teaching Solutions.

Bull, I. (2009) *Extension Maths For Higher Achieving – Junior Secondary*. Putney, NSW: Phoenix Education Pty Ltd.

Bull, I. (2009) *Extension Maths For Higher Achieving – Middle Secondary*. Putney, NSW: Phoenix Education Pty Ltd.

Bull, I. et al (2006) *Maths Dimensions 7 essential learning*. Pearson Education

Bull. I. Maths *Practice and Tests*, 7&8. Putney, NSW: Phoenix Education Pty Ltd.

Bull. I. (1995) *Mathematics Revision and Practice.* Putney, NSW: Phoenix Education Pty Ltd.

de Bono. Edward (2007) Edward de Bono's Thinking Course. *Powerful tools to transform your thinking*: BBC active, England.

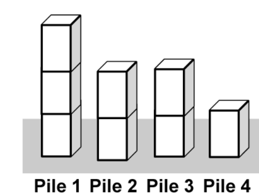
Appendix

SECTION 8: Sharing and describing items

CLASS ACTIVITY

1. Whole class activity/discussion

Write clues so that the person sitting next to you can write the number of cubes in each pile – it is important to get the correct number of cubes in each pile.



2. Pair Activity

Divide students into pairs and give each student one card at random from page XX, face down. Each student needs to write clues for the pile of cubes on their card. The number of cubes in any pile cannot be given in a clue. The students swap cards and each student draws the cubes in piles using the clues.

SECTION 8: Sharing and describing items
LESSON TASKS

1. Jean and Peter share $21 so that Jean gets twice as much as Peter. How much does each person get?
2. Jo is looking at three piles of cubes. There are 12 cubes in total. The number of cubes in the first pile is one more than the number in the second pile and the second pile has one more cube than the third pile. Draw the cubes in the correct piles.
3. Bianca has 15 cubes which she stacks into four piles. Three piles have the same number of cubes. The piles have either 3 or 4 cubes in them. Find the number of cubes in the piles.

SECTION 8: Sharing and describing items
APPLICATION TASKS

1. Ali, Bree and Carla share a prize of $35 won at the local school fete. Ali got $5 more than Bree. Carla got $20 and Bree got $5. Find how much money each person got.
2. Allan, Biri and Cleo need to share $35. Allan gets twice as much as Cleo but half as much as Biri. Find how much money each person got.
3. Zoe has 15 cubes which she stacks in four piles.
(a) Find the number of cubes in the piles if three piles have the same number of cubes. The piles have either 3, 4 or 5 cubes in them.
(b) Find the number of cubes in each pile if the number of cubes in the 4th pile is twice that of the 2nd pile. The number of cubes in the 2nd pile is 2 more than the 1st pile and there are 4 more cubes in the 3rd pile than the 1st pile. There are 5 cubes in the 3rd pile.
4. Henry has 25 cubes. He puts some of them into four piles so that the number of cubes in the 1st pile is 3 times the number in the 3rd pile and the number in the 4th pile is twice the number in the 3rd pile. There is 1 more cube in the 2nd pile than the 3rd pile. List the ways that he can do this.

SECTION 8: Sharing and describing items
EXTENSION TASK 2

1. Four friends share a lottery win. Jane gets twice as much as Jim. Ji gets one dollar less than Jim and Jack gets two more dollars than Ji. If one of the friends gets $14 find the amount that each person receives and find the lottery win.
2. Four friends share a cash prize at the school raffle. Chris gets half as much as Tim and Colin gets half as much as Chris. Matt gets five times as much as Colin. If one of them got $20 how much did the other friends get and what was the total prize?
3. Counters are placed into four piles so that the number in the first pile is twice the number of counters in the third pile. The number in the fourth pile is three times the number of counters in the third pile. There are five more counters in the second pile than the first pile. Find the number of counters in each pile and the number of counters left over if the total number of counters used was
4. (a) 23 (b) 89 (c) 120 (d) 182

I have acknowledged the de Bono book in the references.
Footnotes for page 1
1. *Edward de Bono's Thinking Course – BBC Active*, UK. Page 13
2. *Edward de Bono's Thinking Course – BBC Active*, UK. Page 67
3. *Edward de Bono's Thinking Course – BBC Active*, UK. Page 13
4. *Edward de Bono's Thinking Course – BBC Active*, UK.
5. *Edward de Bono's Thinking Course – BBC Active*, UK. Page 55

TEACHING STUDENTS 'AT RISK': HUME NUMERACY INTERVENTION PROGRAM

Mark Waters and Pam Montgomery

Hume Region, Department of Education and Early Childhood Development, Victoria

"Amie has so much trouble understanding maths that I just get her to memorise, or follow rules. I know that's rote learning, but I don't know what else to do." Year 6 teacher

"Sally has a lot of 'gaps' in maths. I've tried giving her extra worksheets so she can catch up, but she still seems confused." Year 3 teacher

"Carl really struggles in maths. I tried putting him on 'X' computer program, but all he did was hit keys until he fluked the right answers. I wish I knew how to teach those early maths strategies that he's missed." Year7 teacher

Do these teachers' comments sound familiar? This article describes a regional approach to assisting classroom teachers across P-8 to better help their students 'at risk' in mathematics.

Context

Hume Region is the government school region situated in the north-east of Victoria. Across the region, a fundamental problem in numeracy was evident: most classes from P-8 had several students 'below standard' in numeracy, and many teachers felt unsure of how to assist these students.

There was a need to build teacher capacity to assist 'at risk' students in numeracy, whilst simultaneously improving the students' achievement. A program applicable from P-8 that focused on both student and teacher was developed.

The resulting program – the Hume Numeracy Intervention Program (HNIP) – has been operating across the region throughout 2010. There are two components:

- the HNIP framework for implementing numeracy intervention in schools, and
- HNIP teacher training.

HNIP Framework for Numeracy Intervention

The HNIP framework provides a consistent structure for schools to use when providing numeracy intervention. This framework enables the classroom or mathematics teacher to undertake customised teaching for an 'at risk' student from their class, with support from the school. The framework has the following elements:

1. Diagnostic Student Assessment
2. Targeted Maths Plans
3. Differentiated Classroom Lessons
4. Additional Targeted Teaching
5. Daily Independent Practice

1. Diagnostic Student Assessment. The teacher identifies students 'at risk' in numeracy from their class. One of these students is nominated as a Case Study Student. The Case Study Student is assessed by the teacher using a one-on-one interview assessment. The assessment tool used is the Number Fluency Assessment (Waters & Montgomery, 2005). From this assessment, the teacher can identify the student's strengths and their near needs in Number and Structure.

2. Targeted Maths Plans. The teacher videorecords their interview assessment of the student, and then reviews this with the Number Fluency Assessment student record booklet. From these two data sets, the classroom teacher develops a Targeted Maths Plan for the student. This involves collating the student's strengths, identifying their near needs, and setting short term and long term goals for targeted teaching and learning in Number and Structure.

3. Differentiated Classroom Lessons. The student participates in all class mathematics lessons, but the teacher differentiates each lesson so the student understands the mathematical content. This is to ensure that each mathematics lesson is within the student's zone for learning. This may be achieved by the teacher using open-ended tasks, and/or by using

concrete and visual aids as learning supports, and/or by modifying the number range for the student, and/or by using small group instruction within the class lesson.

4. Additional Targeted Teaching. The teacher provides one extra half-hour of one-on-one teaching per week for the student. This is provided for at least ten weeks. Within the half-hour, the student is taught two specific concepts, each one embedded within the context of a task from the Number Fluency Task Library (Waters & Montgomery, 2005). The student is taught to use particular mathematical strategies and self-checking methods in order to use their allocated Number Fluency Tasks independently. Each half-hour of Additional Targeted Teaching follows a set outline of: Familiar Task, Tweaked Task, New Task, and Negotiation of Independent Practice.

5. Daily Independent Practice. After Additional Targeted Teaching (see above), the student uses their Number Fluency Tasks to practise the new strategies. Number Fluency Tasks differ from conventional practice tasks. They are repeatable 'hands-on' tasks that use materials such as playing cards, dice, and game boards. They are customised to closely match the level of the student's learning, and incorporate models and self-checking devices so the student can self-correct any errors. Because of these design features, the tasks can be 'driven' and monitored completely by the student. Once the task is introduced within the Additional Targeted Teaching session, the student is then provided with 20-30 minutes each day at school to independently work on their assigned Number Fluency Tasks. The aim is to build self-monitored 'deliberate practice' so the student fluently uses the taught strategy (Hattie, 2009, p. 185). The student also develops independent learning skills in this time.

Teachers implement the HNIP framework with their nominated student as they participate in HNIP teacher training. All components of the HNIP framework are implemented by the teacher for the Case Study Student during the intervention period (usually ten weeks). The interplay between differentiated classroom lessons, additional targeted teaching, and independent practice is essential for accelerating the student's mathematics achievement.

HNIP in Action

To understand what this looks like in practice, let's follow a Case Study Student and his teacher:

Ryan, a Year 5 student, transferred into school mid-way through first term. His teacher quickly realised that he struggled with many aspects of mathematics. Although he stayed 'on task' during lessons, Ryan took a long time to complete problems. He often counted by ones to calculate, sometimes drawing many tiny lines on his page and counting these up.

Through participating in HNIP training, Ryan's teacher was able to implement the following program:

1. Diagnostic Student Assessment. Ryan's teacher conducted the Number Fluency Assessment as a one-to-one mathematics interview. This enabled Ryan's teacher to record his understandings in Number and Structure, and the range of strategies that he employed to solve problems.

2. Targeted Maths Plans. From the assessment videorecording and student booklet, Ryan's teacher was able to pinpoint his strengths and near needs, and develop a Targeted Maths Plan with customised long and short term learning goals in Number and Structure.

3. Differentiated Numeracy Lessons. Ryan's teacher could now consider his near needs when planning classroom mathematics lessons. With Number and Structure tasks, she modified the number range for Ryan. She thought carefully about appropriate concrete and visual supports, such as having Ryan use a 'jump' method on an open number line when adding and subtracting. With a clear knowledge of Ryan's needs, she was able to teach him more appropriate mathematics strategies within the class lesson.

4. Additional Targeted Teaching. Ryan's school provided coverage for his teacher to take the half-hour of Additional Targeted Teaching each week for a ten week period. Through the HNIP training, Ryan's teacher learned about specific strategies and models, such as using arrays to teach multiplication. Using a range of Number Fluency Tasks as contexts, Ryan's teacher taught him the following strategies:

- Addition and subtraction strategies within 20: near doubles, fact families, and bridging across 10
- Using an open number line for addition and subtraction of two digit numbers
- Using front end methods for addition of two digit numbers
- Skip counting by 3s, 4s and 9s. Relating these to other known skip counting sequences (e.g. 4s is double the 2s, 3s is 2 and 1 more, 9s is 10 more subtract 1)
- Using skip counting to multiply by 10s, 5s, 2s, 3s, 4s, and 9s
- Making and reading arrays to figure multiplication and division (10s, 5s, 2s, 3s, 4s, and 9s)
- Using fact families for division (10s, 5s, 2s, 3s, 4s, and 9s)
- Using build up and build down strategies to multiply (e.g. using 'build down' to multiply a number by 9: think multiply the number by 10, then subtract 1 lot of the number)

An example of one of Ryan's Additional Targeted Teaching sessions is included as *Figure 1*.

5. Daily Independent Practice. Ryan had two Number Fluency Tasks to work on each week. These were the tasks taught by his teacher during the Additional Targeted Teaching time. Ryan practised one task during class time each day, when all students were involved in hands-on fluency practice tasks. He practised the other task at 8:45 each morning, coming early into class and working independently. The tasks built Ryan's understanding and fluency in using specific strategies. Each week his teacher changed or extended the tasks in response to Ryan's progress.

At the end of the ten week program, Ryan's teacher re-assessed Ryan using the Number Fluency Assessment. Ryan's progress equated to advancing one VELS level in both Number and Structure within ten weeks (VCAA, 2007).

HNIP Additional Targeted Teaching Session

Student Name: Ryan **Teaching Session #:** 5

Task	Observations of Student Learning
1. Familiar Task Skip counting by 4s: Review **Predict-a-count** task using calculator: forward by 4 Think-aloud: • Use counting by 2s to work out counting by 4s – skip a 2s number, say the next 2s number	**Notes on student's attempts, thinking** Fluent to 48. Self-correcting any errors Ready to use with array cards to multiply by 4s
2. Tweaked Task Using arrays to multiply: Review **Name That Array** task: 5s array cards, transfer to 3s array cards Think-aloud: • Describing arrays – rows of 3 • Skip count up by 3s to find the product	**Notes on student's attempts, thinking** Quickly saw how to name the array as e.g. 6 x 3 Used skip counting to find product for 4, 6, 8, 7, 9 rows of 3 10, 1, 2, 3, 5 rows of 3 are immediate known facts Used self-checker (product on reverse of array card) well to monitor predictions
3. New Task Develop front-end addition of two-digit numbers: **What Decade Is It?** task using playing cards Think-aloud: • Language of e.g. 28 + 54: The tens make to 70, but the ones go over a ten, so it's in the 80s Record as e.g. 28 + 54 → in the 80s	**Notes on student's attempts, thinking** 45 + 38 → in the 80s 26 + 35 → in the 60s 39 + 58 → in the 90s 57 + 29 → in the 80s 17 + 57 → in the 70s 26 + 48 → in the 70s Took a while to get the language under control Next week tweak to do this task with accuracy in units as well as tens
4. Negotiation of Practice and Record Keeping Practise: 1. **Name That Array** task: 3s array cards 2. **What Decade Is It?** task using playing cards	**Notes on agreed practices** 10 mins morning practice before school each day 10 mins during maths class each day

Figure 1: An Additional Targeted Teaching Session

HNIP Teacher Training Program

HNIP teacher training is available to any teacher of mathematics in Hume Region from P-8. The training consists of five afternoons distributed across one school term (ten weeks), with follow-up case study work at school. Though drawn from evidence-based research, the training is largely practical and action-based.

The teacher implements the HNIP framework for the duration of the training: that is, they work with a nominated student 'at risk' in numeracy from their class. The teacher provides one half-hour of Additional Targeted Teaching for that student each week, and sets up a routine of daily independent practice for the student. During training, the school provides the participating teacher with one hour per week to prepare for, and undertake the Additional Targeted Teaching with the student.

HNIP training aims to develop high quality teaching that helps the 'at risk' student to understand mathematical concepts and strategies. Teachers learn to teach concepts and flexible strategies for Number and Structure. This is based on the belief that:

> "Low achieving students who struggle to master more and more procedures, without using numbers flexibly or compressing concepts, are working with the wrong model of mathematics. These students need to work with someone who will change their world view of mathematics and show them how to use numbers flexibly and how to think about mathematical concepts." (Boaler, 2008, p. 152).

Teachers are taught to assess a student's mathematical understandings, and to customise a Targeted Maths Plan for that student. They are then trained to select and use mathematics tasks that are 'hands-on', able to be driven and monitored by the student, and matched to the student's learning zone. High effect scaffolding techniques are practised, such as using 'think-aloud' when instructing the student (NCTM, 2007). The program also trains teachers to develop connected sequences of lessons that are customised for the student.

Within the training, participating teachers are provided with aligned resource materials. This includes the diagnostic Number Fluency Assessment tool, pathways for plotting student learning trajectories in Number and Structure, and a library of 'hands on', strategy-based Number Fluency Tasks (Waters & Montgomery, 2005). Provision of quality materials helps to ensure that teachers do not use nonsense materials such as worksheets, exercise pages, or computer 'drill' programs with their students.

A key feature of HNIP training is that all participating teachers regularly film their Additional Targeted Teaching sessions. These are reviewed and evaluated by participating teachers at subsequent training sessions. This enables teachers to discuss and improve the quality of their teaching moves, and the precision of their task 'matching' to student needs.

Once a teacher has completed HNIP training, elements of the framework are integrated into their classroom practice. This means that the Additional Targeted Teaching is now undertaken within the teacher's regular mathematics lessons. To do this, most teachers split the half-hour format into smaller sessions distributed across the weekly mathematics lessons.

Initial Findings

Both qualitative and quantitative data have been collected throughout 2010. Data are currently being collated and interpreted. Over 200 teachers from 70 schools have trained (approximately 150 primary and 50 secondary teachers). Over 200 case study students have been involved from P-8. Some schools have had just one or two teachers participate; some schools have had a key teacher train at each year level. Some smaller schools have had all their classroom teachers participate in HNIP training. Early findings indicate:

Case Study Students

- All Case Study Students make accelerated progress in Number
- All Case Study Students improve their independent learning skills
- Well designed Number Fluency Tasks with self-checking mechanisms enable students to practise and monitor their mathematics independently
- Daily independent practice is essential for student progress
- Daily independent practice requires a strong routine to be set and maintained by the teacher

Teachers

- Teachers value training in 'how to teach' specific mathematical concepts and strategies
- Teachers require access to quality sequenced materials for numeracy intervention
- Through closely working with a Case Study Student, teachers' expectations for the numeracy learning of 'at risk' students are raised
- Regularly filming and reviewing one's own teaching of mathematics via digital video is a powerful way to improve the quality of teaching

Schools

- Schools need to prioritise both additional targeted teaching and independent practice for 'at risk' students
- Schools need to provide teachers with time to undertake training and case study work
- Principal involvement in the program has produced more consistent teacher support

Future work for Hume Region for 2011 is to continue HNIP teacher training to reach a greater number of teachers across P-8, to continue to monitor the achievement of Case Study Students, and to research how HNIP trained teachers are able to integrate HNIP practices within their regular mathematics lessons.

References

Boaler, J. (2008). *What's Math Got To Do With It?* Penguin: New York.

Hattie, J. (2009). *Visible Learning: A Synthesis of Over 800 Meta-analyses Relating to Achievement.* Routledge: Oxon.

NCTM Research Brief (2007). *Effective Strategies for Teaching Students with Difficulties in Mathematics.* http://www.nctm.org/uploadedFiles/Research_News_and_Advocacy/Research/Clips_and_Briefs/Research_brief_02_-_Effective_Strategies.pdf

Victorian Curriculum and Assessment Authority, 2007. *Victorian Essential Learning Standards: Mathematics.* http://vels.vcaa.vic.edu.au/vels/maths.html

Waters, M. & Montgomery, P. (2005). *Number Fluency Task Library.* Unpublished document.

Waters, M. & Montgomery, P. (2005). *Number Fluency Assessment.* Unpublished document.

QUADRILATERAL QUARRELS

Allan Turton

Origo Education

An analysis of curriculum documents and standards over recent decades indicates that the study of quadrilaterals in the K-10 years has not been explicitly or consistently explained. This paper outlines a hierarchy to clarify the relationships between the quadrilaterals that are commonly encountered in those year levels.

Classification

As an introduction to examining quadrilaterals, consider the following statements and pictures:

a. "A woman is a human."
b. "Sort reptiles, insects and humans (including women) into groups."
c. "Identify any special living organisms which are mammals (humans, including women)."
d. "Draw pictures of reptiles, insects, women and humans."
e. "Analyse a picture of a vertebrate to determine whether it is a human or a mammal."
f.

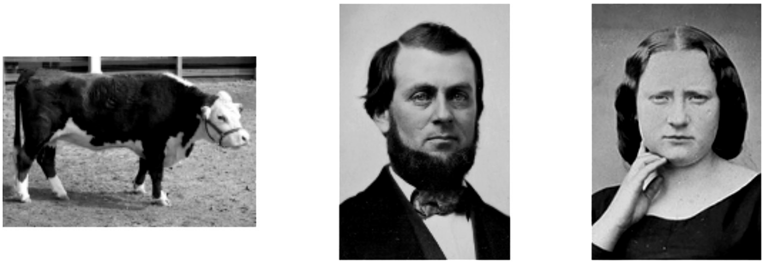


Figure 1. Types of vertebrates. From left: mammal, human, woman.
[Photo source: Flickr (2010a, 2010b, 2010c)]

Statements (a) and (b) make clear the links between women and humans: women are a subset of humans. Statement (c) notes that both humans and women are mammals despite the suggestion in statement (d) that women really are quite distinct from humans. However, statement (e) asserts that humans cannot really be mammals at all, a point which is emphasised by Figure 1. Humans and women clearly do not look like the representative mammal and there is now pictorial evidence that women are not humans although they share some features with them.

It is obvious that when taken as whole, the statements above are not only illogical but also terribly insulting. Sadly, analysis of curriculum documents from the past few decades showed many similar flaws in relation to quadrilaterals:

a. "...a square is a rectangle" (Department of Education Queensland (DEQ), 1968, p.240)
b. "...sorting circles, triangles and rectangles (including squares) into groups" (Australian Curriculum, Assessment and Reporting Authority (ACARA), 2010, Year 2)
c. "...identify any special shapes which are parallelograms (rectangles, including squares)." (DEQ, 1989, p.170)
d. "...construct circles, triangles, squares and rectangles" (Australian Education Council & Curriculum Corporation (Australia), 1991, p.89)
e. "...measures angles in a quadrilateral to determine whether it is a rectangle or a parallelogram" (Board of Studies NSW (BOSNSW), 2002, p.185)
f.

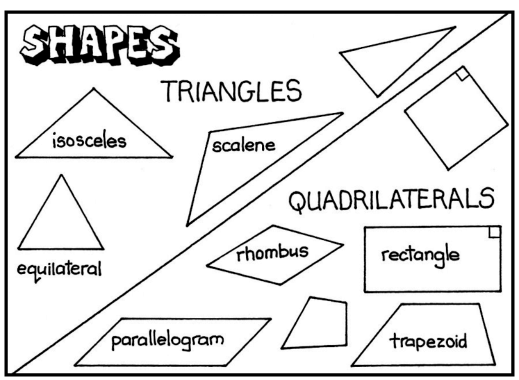


Figure 2. Identification chart (DEQ, 1989, p.112).

By themselves, and with certain definitions in mind, each statement and picture could be seen as accurate. Read together, however, they display many of the same absurdities as the statements about humans.

What is the problem?

Many of the documents that were analysed expressed the view that mathematics is about reasoning and relationships. In regards to quadrilaterals, this was often highlighted by encouraging the organisation of "families" of shapes with increasing levels of specificity. This is analogous to grouping living organisms into general categories to create taxonomies. Figure 3 below shows a small section of such a taxonomy and it can be seen that birds and dogs are both vertebrates (and so also animals), but of these two vertebrates only dogs are mammals.

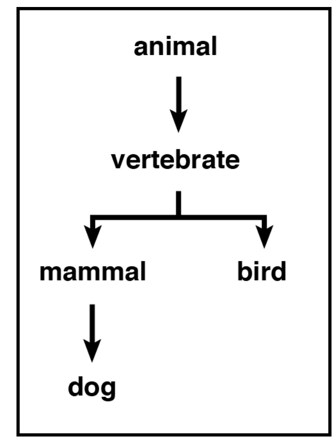


Figure 3. Sample taxonomy

In these types of biological taxonomies, every "end-point" of the family tree has a name (such as "dog" or "bird"). Items at the higher levels of the hierarchy are more inclusive (have very general criteria for membership) while those near the bottom are more exclusive (have very specific criteria for membership). With quadrilaterals, part of the process of creating the family trees is impeded by the fact that there are not enough names to label all the end-points, so that there is a doubling-up of meaning for particular terms. To illustrate this, two statements are taken from the 2002 New South Wales K-6 syllabus. They read:

> "...parallelograms, which are inclusive of the rectangles..." (BOSNSW, 2002, p.117)
>
> "...measures angles to determine whether it is a rectangle or a parallelogram" (BOSNSW, 2002, p.185)

In this example the two shapes appear to take on different meanings in each statement. The first statement states that rectangles are types of parallelograms. This was a fairly common approach in the documents that were reviewed. Usiskin, Griffin, Witonsky and Wilmore (2008, p.26) argue that (at least in textbooks) there is "no disagreement... regarding which special types of quadrilaterals are always parallelograms", naming rectangles, rhombuses and squares as these special types. However, the second statement from the NSW syllabus implies that rectangles and parallelograms must actually be separate classes of shapes. Presumably, what is meant in the second statement is a comparisons between non-rectangular parallelograms and rectangular ones. The absence of any name, let alone a succinct one, for non-rectangular parallelograms creates confusion between seeing parallelograms as an inclusive "family" and seeing them as an exclusive "end-point".

The 2002 New South Wales syllabus is not the only document where this confusion occurs. However, the fact that any definition or description of class inclusion for quadrilaterals is given should be applauded. Approximately 40 documents from the 1960s onwards were examined for this paper, and included Australian national statements and standards, together with K-10 syllabuses and their support documents from New South Wales, Queensland and Victoria. Despite many statements declaring that families of quadrilaterals were to be identified by students, only about a quarter of those documents gave any definitions of the types of quadrilaterals they refer to and even less described the relationship between them. Three had glossaries: the 2002 New South Wales K-6 syllabus (BOSNSW, 2002), the Victorian 2008 VELS (Victorian Curriculum and Assessment Authority, 2008) and the 2010 draft Australian Curriculum (ACARA, 2010). However, only the New South Wales syllabus actually had definitions for the quadrilaterals that were mentioned in the body of the document. The National Numeracy Review recently recommended that "the language and literacies of mathematics be explicitly taught by all teachers of mathematics" (Council of Australian Governments, 2008, p.3). Given the paucity of information on quadrilaterals provided in syllabus documents, including the 2010 draft Australian Curriculum, it may be that there is a need for the language of mathematics to be explicitly taught not just by teachers, but to teachers as well.

What would help?

Two recommendations that are given to help students develop robust and accurate understanding of two-dimensional shapes is to use a wide range of examples (together with non-examples) and explicitly identify which attributes are key to defining the shapes and which are not (Clements & Sarama, 2000; Clements, Swaminathan, Hannibal & Sarama, 1999; Fuys & Liebov, 1997; Sarama & Clements, 2009). However, these recommendations will arguably be much harder when terminology switches back and forth between meanings as in the parallelogram example given previously. Figure 4 below illustrates one possible way of organising quadrilaterals into a hierarchy. The "endpoints" contain a number of new terms whose origins are explained further below.

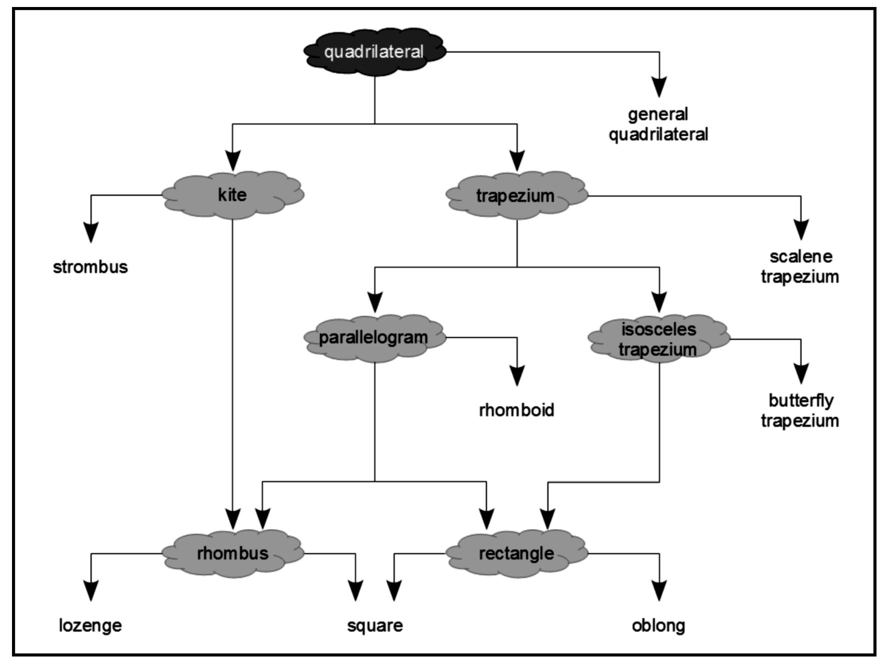


Figure 4. Hierarchy of quadrilaterals

(See BOSNSW (2002), Craine & Rubenstein (1993), De Villiers (1994), Origo Education (2007) and Usiskin et al. (2008) for similar examples)

The larger "families" of quadrilaterals are shown in the "clouds", while the "end points" are shown with bare text. For example, an oblong is a type of rectangle, which is in turn a type of parallelogram, which is a type of trapezium and so on. A rectangle, however, also has some properties of isosceles trapeziums and so can be placed in that family as well. Table 1 lists properties of the quadrilaterals to highlight the relationships between the shapes (note: these are not necessarily minimum properties).

	Pairs of parallel sides	Pairs of equal adjacent angles	Pairs of equal opposite angles	Pairs of equal adjacent sides	Pairs of equal opposite sides
Quadrilateral	may have	may have	may have	may have	may have
Kite	may have	may have	at least 1	at least 2	may have
Trapezium	at least 1	may have	may have	may have	may have
Parallelogram	exactly 2	may have	exactly 2	may have	exactly 2
Isosceles trapezium	at least 1	at least 2	may have	may have	at least 1
Rhombus	exactly 2	may have	exactly 2	exactly 4	exactly 2
Rectangle	exactly 2	exactly 2	exactly 4	may have	exactly 2
General quad-rilateral	none	no more than 1	no more than 1	no more than 2	no more than 1
Strombus	none	none	exactly 1	exactly 2	none
Scalene trapezium	exactly 1	no more than 1	none	no more than 1	none
Butterfly trapezium	exactly 1	exactly 2	none	no more than 2	exactly 1
Rhomboid	exactly 2	none	exactly 2	none	exactly 2
Lozenge	exactly 2	none	exactly 2	exactly 4	exactly 2
Square	exactly 2	exactly 4	exactly 2	exactly 4	exactly 2
Oblong	exactly 2	exactly 4	exactly 2	none	exactly 2

Table 1: Properties of quadrilaterals.

Origins of uncommon terms

These terms were chosen because they were brief, clear or had a precedent but the point is not that these terms are "correct" or the most "appropriate" choices. It is that if an inclusive approach to definitions is used, then having a means of describing shapes without contradiction is needed. These terms provide a means.

Strombus

Conway (2000), coined this term from the Greek word for a spinning top, some of which have the same general profile as this quadrilateral. A type of mollusc also shares this name and has a similar overall profile.

Scalene trapezium

"Scalene" is used in the same sense that "isosceles" is used through analogy with triangles. So in a scalene trapezium the non-parallel sides are different lengths, meaning that "right trapeziums" would also be included. The term has been in use for at least 100 years (see Battista, 2007, p.860; Goodspeed, 1903, p.134)

Butterfly trapezium

Whiteley and Moshe (2005) proposed this term for the isosceles trapezium that has exactly one pair of parallel sides, based presumably on its shape. An alternative is "oblique isosceles trapezoid" (Groves, 2010).

Rhomboid

Heath's (1956) translation of the term used to describe this type of parallelogram was essentially the same as Euclid's original Greek (circa 300 BC). It is described as "that which has its opposite sides and angles equal to one another but is neither equilateral nor right-angled" (Fitzpatrick, 2008; Heath 1956).

An alternative of "oblique parallelogram" has been used at times, however the term does not sufficiently distinguish between lozenges and rhomboids to be a precise description (Department of Education New South Wales, 1967; National Committee on Mathematical Requirements, 1921).

Lozenge

As described by Weisstein (2010b),

An equilateral parallelogram whose acute angles are 45 degrees. Sometimes, the restriction to 45 degrees is dropped, and it is required only that two opposite angles are acute and the other two obtuse.

It is in the latter sense that this term is used in the hierarchy above. An alternative is "oblique rhombus" that follows on from other uses of the word oblique to describe non-perpendicular things (Groves, 2010). Another alternative often used is "diamond", although this term can be related to orientation only (i.e. diagonals are horizontal and vertical) so that when the shape is rotated it is no longer seen as a diamond. The fact that in everyday use a diamond can refer to either a lozenge (like on playing cards) or a square (as seen in a baseball diamond) indicates that its meaning is actually that of "rhombus". (Groves, 2010; Weisstein, 2010a).

Oblong

In Heath's (1956) translation of Euclid's "Elements", oblong is used, a term derived from Latin, which stands in contrast to the use of "rhomboid" noted above. The original Greek was "eteromikes" and is described as being "that which is right-angled but not equilateral" (Fitzpatrick, 2008, p.7; Heath, 1956; OED, 2010).

Conclusion

Terminology for quadrilaterals has been and remains confused. Usage is not consistent and does not encourage an understanding of the relationships between the different types of quadrilaterals. Arguably, this is partly caused by a lack of terminology that is able to separate "family" names from "end-point" names. If quadrilaterals are to continue being a part of K-10 mathematics, as they are set to be with the 2010 draft Australian Curriculum, then perhaps more consideration should be given to the language that students are expected to learn to ensure it reflects the logical nature of mathematics.

References

Australian Curriculum and Reporting Authority. (2010). Australian Curriculum: Mathematics: Draft Consultation version 1.0.1. Retrieved 14 August, 2010, from http://www.australiancurriculum.edu.au/Explore.mvc/Mathematics

Australian Education Council & Curriculum Corporation (Australia). (1991). *A national statement on mathematics for Australian schools: A joint project of the States, Territories and the Commonwealth of Australia.* Carlton, Vic.: Curriculum Corporation.

Battista, M.T. (2007).The development of geometric and spatial thinking. In Lester, F. (Ed.), *Second Handbook of Research on Mathematics Teaching and Learning: A Project of the National Council of Teachers of Mathematics* (pp. 843-908). Reston, VA: National Council of Teachers of Mathematics.

Board of Studies New South Wales. (2002). *Mathematics K – 6 syllabus*. Sydney, NSW: Author.

Clements, D.H., & Sarama, J. (2000). Young children's ideas about geometric shapes. *Teaching Children Mathematics,* 6(8), 482-488.

Clements, D.H., Swaminathan, S., Hannibal, M.A.Z., & Sarama, J. (1999). Young children's concepts of shape. *Journal for Research in Mathematics Education,* 30 (2), 192-212.

Conway, J. (2000, May 18, 12:21 pm). *Re: quadrilateral.* Message posted to http://mathforum.org/kb/thread.jspaforumID=128&threadID=353167&messageID=1080580#1080580

Council of Australian Governments Human Capital Working Group. (2008). *National numeracy review report / Commissioned by the Human Capital Working Group, Council of Australian Governments.* Canberra, ACT: Author.

Craine, T.V., & Rubenstein, R.N. (1993). A quadrilateral hierarchy to facilitate learning in geometry. *The Mathematics Teacher,* 86(1), 30-36.

De Villiers, M. (1994). The role and function of a hierarchical classification of quadrilaterals [Electronic version]. For the learning of mathematics, 14(1), 11-18.

Department of Education New South Wales. (1967). *Curriculum for primary schools: Mathematics.* Sydney, NSW: Government Printer.

Department of Education Queensland. (1968). *A program in mathematics for primary schools: Stages 7 - 8.* Brisbane, QLD: Government Printer.

Department of Education Queensland. (1989). *Years 1 to 10 Mathematics Sourcebook: Activities for teaching mathematics in Year 6.* Brisbane: Author.

Fitzpatrick, R. (2008). *Euclid's elements of geometry*. Retrieved 14 August, 2010, from http://farside.ph.utexas.edu/euclid.html

Flickr (2010a). *NMA.0052946_01*. Retrieved 14 August, 2010, from http://www.flickr.com/photos/34380191@N08/4677707437/

Flickr (2010b). Rose, Polled *Hereford Cow.* Retrieved 14 August, 2010, from http://www.flickr.com/photos/nostri-imago/3016910852/

Flickr (2010c). *Unidentified man.* Retrieved 14 August, 2010, from http://www.flickr.com/photos/boston_public_library/2402803821/

Fuys, D., & Liebov, D. (1997). Concept learning in geometry. In D. Chambers (Ed.) (2002) *Putting Research into Practice* (pp. 156-159). Reston, VA: National Council of Teachers of Mathematics.

Goodspeed, E. J. (1903). The Ayer papyrus [Electronic version]. *The American Mathematical Monthly,* 10(5), 133-135.

Groves, J. (2010, March 15, 12:52 pm). *Re: Inclusive and exclusive definitions... again! Message* posted to http://mathforum.org/kb/thread.jspa?forumID=206&threadID=2050392&messageID=7012073#7012073

Heath, T.L. (Ed.). (1956). Euclid, Elements. Retrieved 14 August, 2010, from

http://www.perseus.tufts.edu/hopper/text?doc=Euc.+1&redirect=true

National Committee on Mathematical Requirements. (1921). *The reorganization of mathematics in secondary education: A summary of the report by the national committee on mathematical requirements.* Washington, DC: Government Printing Office.

Origo Education. (2007). *The Origo handbook.* Brisbane, Qld: Origo Education.

Oxford English Dictionary. (2010). *Oblong*. Retrieved 14 August, 2010, from

http://dictionary.oed.com

Sarama, J., & Clements, D.H. (2009). *Early Childhood Mathematics Education*

Research: Learning Trajectories for Young Children. New York: Routledge.

Usiskin, K., Griffin, J., Witonsky, D., & Willmore, E. (2008). *The classification of quadrilaterals: A study of definition.* Charlotte, NC: Information Age Publishing, Inc.

Victorian Curriculum and Assessment Authority. (2008). *Victorian essential learning standards discipline-based learning strand: Mathematics* (Rev. ed.). Retrieved 16 August, 2010, from http://vels.vcaa.vic.edu.au/downloads/vels_standards/

velsrevisedmathematics.pdf

Weisstein, E.W. (2010a). *Diamond*. Retrieved 14 August, 2010, from

http://mathworld.wolfram.com/Diamond.html

Weisstein, E.W. (2010b). *Lozenge*. Retrieved 14 August, 2010, from

http://mathworld.wolfram.com/Lozenge.html

Whiteley, W., & Moshe, L. (2005). *Making Sense of Transformations and Symmetry - the Heart of Geometry.* Retrieved 14 August, 2010, from

http://www.dynamicgeometry.com/documents/userGroups/jmm2006/

ExploringParallelorams.doc

ITEMISING ASSESSMENT IN THE SOUTH AFRICAN CURRICULUM GUIDELINES: UNDERSTANDING ISSUES OF TEACHER POWER

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Abstract

This article discusses the general question: who holds "power" in school mathematics education in South Africa? To what extent is the teacher given an opportunity to exercise "power" in mathematics assessment? If teachers are given such power, what does that power allow them to do, and under what conditions does this happen? The case of mathematics is presented here to begin to unpack some complex questions of teacher power in curriculum assessment, to open a discussion about the nature of assessment and to deepen our understanding of the notion of assessment itself. A "work-in-progress" model for understanding assessment is demonstrated in this article. The significance of this "model" is in the metaphorical nature that also has poetic dimensions.

Introduction

From the vantage point of new forms of assessment, this article is an attempt to unpack the question of teacher power by looking at how teachers are positioned in the National Curriculum Statement (NCS) Assessment Guidelines for Mathematics (Grades 10-12, Department of Education (DoE), 2005). I focus on the new assessment guidelines for two reasons. First, it is because it is widely recognised that assessment is the engine of education systems. Conceiving assessment as an engine is a metaphorically powerful way of thinking about education. Stated more practically, when we look at assessment, we look at an engine: what drives[i] education systems. Education systems run continually because they thrive on the fuel of assessment[ii] which has never run dry since its injection into education. The engine-power of assessment can be seen for example, in South Africa, in how the outcomes of assessments are not only celebrated, but also how under-performing schools and their administrators are perceived by society. Second, focusing on assessment is consistent with the view that assessment is an "integral[iii] part of teaching and learning. For this reason, assessment should be part of every lesson and teachers should plan assessment activities to complement learning activities" (DoE, 2005, p. 1)

The DoE (2005) states that guidelines assist teachers in the teaching. Teachers are encouraged to use these guidelines as they prepare to teach the National Curriculum Statement. The assessment guidelines are conceived as a critical resource[iv] that should be able to assist teachers in their teaching of mathematics in accordance with the national policies. Viewing assessment guidelines as a resource i.e. as tools for looking into learning systems (Davis & Simmt, 2003) and what becomes of learning draws us to a key conceptual backbone of educational thinking in our context of education in South Africa in relation to resources and tools for mathematics education. Adler (1999) points out that "access to a practice requires its resources to be 'transparent'" (p. 48). Adler also introduces the notion of "visibility and invisibility" in relation to "transparency in the practice of teaching mathematics" and argues that "Resources need to be seen to be used. They also need to be invisible to illuminate aspects of practice. For talk to be a resource for mathematics learning it needs to be transparent; learners must be able to see it and use it" (p. 63). From a perspective of transparency, I analyse assessment guidelines which mathematics teachers are called upon to use as resources in their work. This analysis attends to the complexity of assessment policies and the legitimating power that they are intended to give to mathematics teachers. In commencing the analysis, I first describe two key aspects that are constituents of the engine of assessment guidelines, namely "daily assessments" and "programme of assessment".

Daily assessment

Two forms of assessment have been proposed in the NCS: continuous[v] assessment and external assessment. Continuous assessment is a form of assessment which when used jointly with "informal daily assessment" and "formal programme of assessment" (p. 1)

is instrumental for: the development of "learners' knowledge, skills and values", and the identification of "learners' strengths and weaknesses". As it stands, continuous assessment should have a significant role to play in shaping learners' learning and "proficiencies" in mathematics. However, given that this form of assessment only "counts 25%" of the final mark at Grade 12, does that not mean that there is less recognition at the policy level of the significance of continuous assessment?

A key component of continuous assessment is "daily assessment" (p. 2). According to the DoE (2005), this kind of assessment is essentially formative as it occurs "during learning activities" where the aim is for the teacher to *monitor* learner progress. Furthermore, it is stated that this monitoring by the teacher "*can* be done through question and answer sessions; short assessment tasks completed during the lesson by individuals, pairs or groups or *homework* exercises" (p. 2, emphasis added). The marking of these assessments has a powerful pedagogical dimension. According to the DoE (2005, p. 2),

> Individual learners, groups of learners *or* teachers *can* mark these assessment tasks. Self-assessment, peer assessment and group assessment actively involve learners in assessment. This ... allows learners to *learn* from and *reflect* on their own performance (emphasis added).

The DoE states that "the results of the informal daily assessment tasks are not formally recorded *unless the teacher wishes to do so*" (p. 2, emphasis added). Nevertheless, there is importance attached to these assessments because "teachers *may* use the learners' performance in these assessment tasks to provide verbal or written feedback to learners, the School Management Team and *parents*". However, given that "the results of these assessment tasks are not taken into account for promotion and certification purposes" puts into question the significance of these assessments.

One might consider these assessment proposals as liberating[vi] given that: a) a range of strategies, not just a single one, are suggested for monitoring learner progress; b) the teacher or learner can mark these assessments, so it does not matter who marks them; c) there is a taken-for-granted assumption that learners should learn from and reflect on their performance as they engage with assessment tasks; and d) "The results of the informal daily assessment tasks are not formally recorded *unless the teacher wishes to do so*". With respect to (a), we need to ask the question: how do teachers decide what form of assessment task should be given to learners and *when*[vii] should this happen? If teachers decide to give learners "homework exercises", how do they *decide* which form of tasks should be allocated for homework? Therefore, while we are told: "teachers' lesson planning should consider which assessment task will be used to informally assess learner progress", it is not clear how the teacher needs to select or plan for these tasks particularly given that there are several forms of regulatory tasks that are seemingly transparently available and made known to teachers. With respect to (b), it is important to ask the question: how are teachers able to decide which tasks should be marked by learners[viii], and which ones can only be marked by teachers? With respect to (c), we need to ask the question: what "opportunities to learn"[ix] (Weber, Maher, Powell & Lee, 2008) mathematics are presented in the tasks and learners' performance in these? How are these learning opportunities evident in tasks? Can teachers anticipate these? In what ways can teachers be able to think about the nature of these opportunities and at what time they might arise? A similar question needing to be asked with respect to (d) is the following: how do teachers decide which assessment results are useful to record and which ones are not? In all these questions lie tensions and dilemmas which undermine the power of teacher decision making because of the contradictory nature in which opportunities to make decisions[x] are framed.

Of pedagogical importance in the NCS guidelines is the importance of feedback. It is stated that "teachers *may* use the learners' performance in these assessment tasks to provide verbal or written feedback to learners, the School Management Team and *parents*. This is particularly important if barriers to learning or poor levels of participation are encountered". Aside from the question of what kind of feedback[xi] is more appropriate and for what purposes, there needs to be engagement with the issue of what kind of feedback needs to be given to parents. In relation to this, how do teachers decide to use verbal rather than written feedback? If written feedback is given to parents particularly the kind of feedback that is consistent with the taxonomy and rating scales proposed (see p. 6 in the NCS mathematics assessment guidelines), how do teachers ensure that parents are able to understand what the feedback means? I ask this question while acknowledging the fact that there does seem to have been a paradigm shift in assessment in South African education that is resonant with the widespread wave of reform that is shaping current theoretical thinking in assessment (Davis & Simmt, 2003), particularly from a complexivist perspective.

It seems quite clear here that teachers have a considerable amount of flexibility in the nature and extent of the assessments that should constitute "daily assessment". However, it is surprising that these daily assessments are accorded very little importance if any at all. According to the DoE, "the results of these assessment tasks are *not taken into account for promotion and certification purposes*" (p. 2). Why should teachers take daily assessments seriously when little value has been placed upon these?

Program of assessment

There is a form of assessment that appears to fall under what is called "Program of assessment" which seems to be more valued than daily assessment.

> Teachers should develop a year-long formal Programme of Assessment for each subject and grade. In Grades 10 and 11 the Programme of Assessment consists of tasks undertaken during the school year and an end-of-year examination. The marks allocated to assessment tasks completed during the school year will be 25%, and the end-of-year examination mark will be 75% of the total mark (DoE, 2005, p. 2).

A key question here is: What is entailed in "tasks undertaken during the school year"? And how much control does the teacher have on the nature of what these tasks should look like? How are these tasks different from "daily assessment" tasks? Whatever these tasks are, it is clear here that because they are developed by the teacher, the teacher has some control over how these need to look like. In fact, because assessment of these tasks "counts 25% of the final grade or year mark", it means that the teacher could take these more seriously than the daily assessments. However, it appears that the teacher has *little control* over the *number* of assessments of this form (Morais, 2002). This is because, according to the DoE (2005, p. 3, emphasis added), "If a teacher wishes to add to the number of assessment tasks, he or she *must* [submit] the changes to the head of department and the principal of the school". In addition, "The teacher must provide the Programme of Assessment to the subject head and School Management Team before the start of the school year". The latter point means that once the teacher has developed the program of assessment, that program is no longer in their control, given that they need to provide a motivation for changing *their own* plan of assessments once submitted to school management, learners and parents.

From the above, there seems to be an emphasis on the "number of assessment tasks" in the Program of assessment, rather than on the nature of those assessments. What is the main reason for asking teachers to submit a plan of assessment to the subject head and the school management team? It is obviously clear that the aim in the NCS guidelines is to ensure that there is a regulatory mechanism that should guide the instrumentation of assessment in schools. However, to what extent does this regulatory mechanism address issues of quality in the way it has been stated? And how would the school management team, learners and parents judge the quality of these assessments?

An interesting development in the NCS assessment guidelines is the fact that there is an attempt to move away from tests and examinations as providing the only means of providing feedback on learners' progress.

> The remainder of the assessment tasks should not be tests or examinations. They should be carefully designed tasks, which give learners opportunities to research and explore the subject in exciting and varied ways. Examples of assessment forms are debates, presentations, projects, simulations, literary essays, written reports, practical tasks, performances, exhibitions and research projects (DoE, 2005, pp. 3-4).

We see here that opportunities are being created, as learners engage with assessments, to "research" and "explore" mathematics as a discipline: what it means, and perhaps how it applies to learners' everyday lives. However, while opportunities are being opened up for assessment, it is not clear what these proposals mean for schools and learners who come from disadvantaged contexts. So the power question here concerns research for what purposes (Murray, 2002) and who benefits from such research.

One robust way in which mathematics can be excitingly explored (performed) is to involve learners in tasks that call upon technological contexts and tools. For example, one assessment standard in Learning Outcome 2 states that we know that learners are able to investigate, analyse, describe and represent a wide range of functions and solve related problems when they are able to

> Generate as many graphs as necessary, initially by means of point-by-point plotting, supported by *available*[xii] *technology*, to make and test conjectures about the effect of the parameters k, p, a and q for functions including: y = sin(kx) (DoE, 2005, p. 16)

In Learning Outcome 4 (data handling), one of the contexts requires learners to calculate "the variance and standard deviation of sets of data manually (for small sets of data) and using *available technology* (for larger sets of data), and representing results graphically using histograms and frequency polygons" (p. 21). Learners are also required to "use available technology to calculate the regression function which best fits a given set of bivariate numerical data" (p. 24). Given the flexibility and efficiency of technologies such as handheld graphing calculators[xiii], the proposals being suggested in the curriculum guidelines are commendable given that they have the potential to allow learners to work efficiently with mathematical ideas and computations involving these. However, while the teacher might plan his/her assessment in keeping with these technological opportunities, one needs to recognise whether in disadvantaged contexts such as rural townships, schools would be able to afford these. In such a case, the *choices* for the teachers are further limited in terms of their selection of assessment tasks and tools that could be used to enhance learners' engagement in these. While technological tools may add a conceptually and didactically powerful dimension to teaching, when the conditions in which teachers teach mathematics are hostile, the power of teaching tools becomes limited.

Some emerging power contradictions

The above analysis of the assessment guidelines has indicated that teachers are given some power and flexibility over what goes on in the daily assessments that learners engage with in their mathematics activities. The teacher is given power to choose from a range of strategies for monitoring learner progress. Once assessment tasks have been undertaken by learners, the teacher can decide whether to mark them or whether learners should mark their own written work. Particularly interesting, the teacher can choose whether to record the results of the assessments or not. "The results of the informal daily assessment tasks are not formally recorded unless the teacher wishes to do so". While it appears that the teacher is given power over assessment at the informal daily level, this power is highly limited for two reasons. First, the results that emerge from the teacher's exercise of such power over assessment are not given much Political significance. Second, it is not clear how the teacher is to exercise such power. Because of these reasons, I propose that while the intention of the NCS is to allow teachers freedom to work in ways they find themselves in their contexts, such freedom is only an imagination. The question then becomes, why should the NCS provide these opportunities for teachers to exercise their freedom or power over assessment when it is known that teachers will eventually have limited power? What is the aim of the NCS in having such proposals? I suggest that the NCS finds itself in this predicament because of an attempt to align itself, as can be expected, to the principles of outcomes-based education, OBE.

According to Spady (1998), there are three key assumptions to OBE. "All students can learn and succeed, but not on the same day in the same way; successful learning promotes even more successful learning; and *schools control the conditions that directly affect successful school learning"* (emphasis added). It is the third assumption that is more pertinent to "blind spots" (le Grange, 2004) and the closed assessment power box I am opening here. It seems that the NCS is attempting to give teachers more power over daily assessment because teachers, as critical constitutive agents of schools, control the conditions that directly affect successful school learning. We are talking here about the day-to-day work of teachers as learning managers[xiv] in their own classrooms. It is the centrality of the teacher that the NCS seems to be rightly uplifting here. According to Todd and Mason (2005), "The most effective factors [for improved learning] depend on the teacher, and other distal variables have an impact to the extent that the teacher exploits their potential in enhancing learning" (p. 229). Todd and Mason continue to suggest that "The challenge for South African teachers is to maximize these proximal factors that have been identified in the research, in spite of the difficulties they face because important distal variables remain unsatisfied". Is the way the NCS assessment guidelines are stated an attempt to satisfy the "proximal" factors associated with effective learning to which the teacher is a central part? The analysis presented above points to the affirmative. I suggest here that a further elaboration of the rationale and conceptualisation of daily assessments is necessary in order for South African education policy to "maximize the ability of teachers to exploit… proximal factors" which according to Hattie (1999, in Todd and Mason, 2005, p. 227) are concerned with teachers coming to "know what our students are thinking so that we can provide more feedback… and develop deep understanding". The key issue centres on recognising the need to have "teachers who understand their discipline well, and who care about their students and what they know". For it is such teachers who "will be better able to set challenging goals and to provide well-directed feedback" (Todd & Mason, 2005, p. 227). I posit that mathematics education in South Africa can only hopefully[xv] be able to obtain such kind of teachers if policies are developed and implemented in such a way that they recognise the power that teachers have over daily assessments in addition to, and more importantly, sensibly recognising the value of these assessments.

References

Adler, J. (1999). The Dilemma of Transparency: Seeing and Seeing through Talk in the Mathematics Classroom. *Journal for Research in Mathematics Education,* 30(1), pp. 47-64.

Adler, J. (2000). Conceptualising resources as a theme for teacher education. *Journal of Mathematics Teacher Education,* 3(3), 205-224.

Davis, B. & Simmt, E. (2003). Understanding learning systems: Mathematics education and complexity science. *Journal for Research in Mathematics Education, 34(2),* 137-167.

Department of Education (DoE) (2005). *National Curriculum Statement Grades 10-12 (General) Subject assessment guidelines (Mathematics).* Department of Education: Pretoria.

Freire, P. (2004). Pedagogy of Hope: Reliving Pedagogy of the Oppressed. New York: Contiuum.

Le Grange, L. (2004). Ignorance, trust and educational research. *Journal of Education, 33,* pp. 69-84

Morais, A. (2002). Basil Bernstein at the Micro Level of the Classroom. *British Journal of Sociology of Education, 23(4),* 559-569.

Murray, S. (2002). Get real? Some thoughts on research for teaching and research for policy. *Journal of Education 27,* 59-78

Sfard, A. (1991). On the dual nature of mathematical conceptions: Reflections on processes and objects as different sides of the same coin. Educational Studies in Mathematics, 22(1), 1-36.

Spady, W., 1998. Outcomes-based education: an international perspective. In: Gultig, J., Lubisi, C., Wedekind, V., Parker, B. (Eds.), *Understanding Outcomes-based Education: Teaching and Assessment in South Africa.* South African Institute for Distance Education and Oxford University Press: Cape Town.

Todd, A. & Mason, M. (2005). Enhancing learning in South African schools: Strategies beyond outcomes-based education. *International Journal of Educational Development 25,* 221–235.

Weber, K., Maher, C., Powell, A., & Lee, H. S. (2008). Learning opportunities from group discussions: warrants become the objects of debate. *Educational Studies in Mathematics, 68,* 247-261.

Endnotes

i drive → dRive → dRiveR → RiveR, DRiver → Diver. Just as there is no drive without a driver, there are no rivers without divers. Rivers make divers. dyvers → DYvErs → DYE. Mathematics education is like a river. Assessment is like a water current (wave) in the river of mathematics. Tests and examinations are dyes that give mathematics the colour it has. In this article, I use a methodology for opening up power questions that involves a re-writing of terms (e.g. drive → DrIvE → DrYvE → DYE; and opportunity → op-port-unity), i.e. a re-presentation. This methodology has a poetic and metaphoric dimension. As I proceed in this way, I distinguish between "representation" and "presentation". Davis (2005) observes that "art has a two-fold function. It both represents (in the sense of calling something to mind, not in the sense of precise or fixed depiction) at the same time that it presents (that is, it opens up new interpretive possibilities)" (p. 24). He further argues that "the two-fold function of representation and presentation – this vital simultaneity – can (and should,) also be a possibility for texts written in standard academic prose" (p. 24). I propose here that re-written terms in the way I have indicated here demonstrates a mode of "presentation" that affords a window into the "opening of new interpretive possibilities" in assessment systems in and about mathematics that transcends conventional forms of "getting to know".

ii How did assessment begin? What instituted the genesis of tests and examinations? And what were education systems like before tests and examinations became instituted in these systems?

iii The term "integral" has at least two meanings, one in language: meaning "being an essential part or intrinsic/constituent part of some entity", and in mathematics (calculus): meaning "the sum of a large number of infinitesimally small quantities, summed either between stated limits (definite integral) or in the absence of limits (indefinite integral) (Collins English Dictionary, 1979). Because of the "openness" in the tone of the policy statements in the new curriculum, one is led to consider assessment not just an integral part of teaching, but more pointedly an "indefinite integral" part of teaching, i.e. teachers are at liberty (have the power) to put in their own limits in their assessments.

iv resource → "re-source". Resource is both a verb (Adler, 2000) and a noun, i.e. resources (such as assessment guidelines) not only need to be available (present) but they also need to be used in order to be seen to be present.

v continuous → co**U**ntinuous → co**U**nt-i-nuo-us → co**U**nt – I – n **U** o – **US**. Continuous assessment is about **US!** Not only does assessment "count" but those involved in assessment also count. Teachers obtain their power in assessment from the perspective that they (teachers) count!

vi "Liberating" implies that teachers know what needs to be done and how.

vii What do teachers understand by the statement: "assessment is an integral part of teaching"? Transparency is both about "when" and "what"; time, content and process (how). [Assessment is an integral part of teaching →... assessment... is... an... integral... **P**art... **O**f... **T**eaching. **POT!** (In-pot → Assessment as a melting **pot** of teaching! Assessment is an integ**RATE**d pot → assessment is rated → assessment has a powerful "rating" in the school curriculum].

viii What goes on in the mind of learners when they are marking peers' written work? Do they see themselves as learners? What identity do they assume?

ix Opportunities to learn need to be visible in order to be seen. What opportunities to learn do learners see? What kind of learners see what kind of opportunities? [Opportunity → Op-port-unity. Assessment not only provides a unifying "pot" for teaching but also a uniting pot (port) for learning].

x To be able to make decisions is to "decide". For teachers to decide, they need to know how to "de-side"**!** That is, they need to be aware of the dual or complementary nature of mathematical concepts and conceptions. Sfard (1991) considers the structural and operational nature of concepts as "different sides of the same coin" (p. 1).

xi To provide feedback means to "feed forward", to inform (i.e. in-**FROM**, to "come in from"). The use of "feedforward" in place of "feedback" in assessment is more forward looking! In looking backwards we do not intend to go backwards but forward!

xii available → avail–able. Available technology implies technology that is available (present) and also able (enables one to engage in mathematical work more efficiently).

xiii Many learners in schools are able to access handheld ordinary scientific calculators but not of a graphical nature. Very few schools have access to the so-called "interactive smartboards". Perhaps we need to revisit our ordinary blackboards in schools so that they could become more "interactive smart **BLACK**boards"**!**

xiv A distinction is made here between teachers as managed agents as opposed to manager agents.

xv The introduction of the new curriculum in South Africa presents opportunities for change in mathematics education in South Africa, at least for future generations! Support for this hope and what one may call "pedagogies of hope" can draw from Freirean perspectives (e.g. see Pedagogy of Hope: Reliving Pedagogy of the Oppressed (2004)). Freire suggests in that volume that "one of the tasks of the progressive educator, through a serious, correct political analysis, is to unveil opportunities for hope, no matter what the obstacles may be. After all, without hope there is little we can do. It will be hard to struggle on, and when we fight as hopeless or despairing persons, our struggle will be suicidal..." (p. 3). This article intends to contribute to a possible window for "unveiling" change possibilities for how assessment is viewed in relation to teachers' mathematical work in the classroom.

HOORAY FOR MARTIN GARDNER!

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Martin Gardner died last May (21-10-1914 — 22-5-2010). I make no claim to be a Gardner expert. But as I regard him as one of my two main mathematics education inspirations (the other is W.W. Sawyer), I want to repay some small part of my debt to Gardner's remarkable career by sharing some of his best games, problems, and puzzles — "his" not because he created them, but because he explained them so brilliantly — and to touch on his Lewis Carroll annotations. Let's begin with polyominoes, the Soma cube, Eleusis, Hex, Phi, ...

Out of *Scientific American* — Mathematical Games and Puzzles

Martin Gardner will be remembered for many years for two important things. The first of these are the many books that were based on his regular column "Mathematical Games and Puzzles" in issues of *Scientific American*. (I have not provided a complete list: I am sure a comprehensive bibliography is easy to find on the internet.) The first of these is *Mathematical Puzzles and Diversions* (1959) — published more than sixty years ago! — yet the book is as fresh as ever. This introduces hexaflexagons, variations on noughts and crosses (including playing in three dimensions), paradoxes of probability, the classic Tower of Hanoi puzzle, curious examples of topology (including a triangulated Moebius strip), Piet Hein's strategy board game Hex, and Solomon Golomb's marvelous polyominoes — plus problems, and puzzles, and reflections on the ambidexterous universe, and a homage to the great American puzzler Sam Lloyd, and more! Through all of this Gardner is, often, merely reporting on, and re-presenting the work of others. Yet despite this, his voice as author is friendly and engaging. And it is obvious that he always goes beyond "mere" reporting, by offering his enthusiasm, critical and curious reflections, and original ideas and examples.

We can appreciate the clarity of expression, focus, and personal originality of Gardner's "mathematics games and puzzle" books if we compare any of them with similar mathematics popularizing, and recreational mathematics books by other writers. One example is David Wells' *Penguin Book of Curious and Interesting Mathematics* (1997). Wells is mathematically able, and a competent writer: for many years he was the puzzle editor for the fascinating British magazine *Games & Puzzles*. But competence and rampantly uncontrolled eclecticism, with an eccentric twist of humour — as I see Wells' book — is no match for Gardner's lucid personable style.

Gardner's second book, *More Mathematical Puzzles and Diversions* (1961) is, naturally, more of the same, the mixture as before, steady as she goes. The Platonic Solids and nets and geometric puzzles, tetraflexagons, the classic British puzzler Henry Ernest Dudeny, Piet Hein's glorious Soma Cube, recreational topology including the network-bridging game Gale, Phi or the Golden ratio, mazes, recreational logic (including grid-type Smith-Jones-Robinson logic puzzles), magic squares, Robert Abbott's game of scientific induction Eleusis, origami, mechanical puzzles (including tangrams, and solitaires), and reflections on probability and ambiguity — plus problems galore, and more!

The later books continue in similar vein. Marvellous stuff! What's not to like?

Importantly, Gardner was always modest in what he did. He saw himself as part of the long chain of others before him who had found delight in curious byways of mathematics, unusual games, teasing puzzles, and sometimes profound ideas. He was a promoter of the great mathematical puzzlers who had gone before him, such as Sam Lloyd, and Henry Dudeny — and he generously promoted his own contemporaries on the other side of the Atlantic, such as T.H. O'Beirne who wrote a similar puzzle-focused column in a British newspaper.

Gardner also attracted others to him, so that he in turn could publicise their work. The games expert and inventor Sid Sackson appeared at times, with news of new games such as the scientific induction game Patterns, and the strategy board game Focus, that uses stackable poker chips to control how a "piece" could move on a square-grid board. Similarly the remarkable British mathematician John Horton Conway was a frequent co-contributor, with his poker-chip cellular automata "The Game of Life", and Sprouts, and his new axioms for the Real Numbers.

Polyiamonds, polyhexes, dominoes, map-coloring, hypercubes, Escher's art, shuffling cards, quilt patterns and rectangular space-filling, random numbers, Pascal's triangle, Fairy (non-standard) chess, Piet Hein's superellipse, trisecting an angle, optical illusions, hyperspheres, random walks, Boolean algebra, teaching computers to think (including a teachable computer made with matchboxes), Turing machines, cyclic numbers, dominoes. Fibonacci and Lucas and Catalan numbers, the abacus, palindromes in words and numbers, Mascheronic constructions (using a "rusty" compass, or only a compass), game theory, acrostics, finger arithmetic, Viking many other board games, polyaboloes, colored triangles and squares and cubes, trees and hacking at graph-theory bushes, dice ...

Occasionally Gardner's column was a more personal exploration of a theme, such as Simplicity, Right and Left, Nothing, Everything, Time Travel. Sometimes he debunked charlatans such as the numerological nonsense-patterns of Freud's colleague Fleiss. Sometimes he played April-Fool's-Day jokes.

Braids, Lewis Carroll's games and puzzles, packing spheres, the fundamental transcendental number Pi, the ellipse, Coxeter, the calculus of finite differences, polycubes, worm paths (spirolaterals), look-see proofs (Behold!), knots, donuts, random music, anamorphic art (such as the famous distorted skull in Holbein's "The Ambassadors"), tiling with convex polygons, maps, space-filling, induction and probability, ...

Importantly, Gardner never wrote a textbook. His discussions are far less structured and formal. They are idiosyncratic, eclectic, and very engaging. Moreover they almost always have great potential as background for stimulating and enriching mathematics activities.

In my opinion every mathematics teacher should personally own as many of Gardner's books as possible, and always keep them handy! From my own experience, when they were prescribed as part of my Secondary teacher-training, I have continually dipped into them. If my own writing has been half as clear and inspirational as Gardner's I will be satisfied, and much of the credit will be due to Gardner!

Gardner and the Annotated Alice, and Beyond!

The second thing Gardner will be remembered for is his wonderful *Annotated Alice* — a scholarly, and insightful commentary on Lewis Carroll's classic "Alice in Wonderland" books. After the valuable Introduction, and annotations on Carroll's prefatory poem "All in the golden afternoon" (Gardner points out that on the day in question, when Carroll and some friends and children went on a rowing picnic, the weather near Oxford was actually cool and rather wet), the first annotation alerts us to the differences between "Alice" as she really appeared, and the way that Carroll's classic illustrator Tenniel drew "Alice". And so on.

But there's more. Gardner also annotated "The Wasp in the Wig", a draft chapter that was omitted from *Through the Looking Glass*: it would have been placed between the departure of the White Knight, and Alice entering the eighth square at the end of chapter

8. And Gardner annotated Carroll's other great nonsense fantasy *The Hunting of the Snark*. And he annotated the "Father Brown" priest-detective stories of G.K. Chesterton.

Although I do not claim to be an expert on annotated books, as far as I know, Gardner's *Annotated Alice* was the first book that provided a running commentary on another author's book, with the possible exception of the extremely erudite commentaries on ancient texts, such as the Old Testament, or Ancient Greek or Roman works. By contrast, "Alice" was commonly regarded as an ordinary, relatively modern book, supposedly so accessible and amusing that it was confidently treated as (merely) a children's book, particularly as it was illustrated, and had been initially improvised as a story invented for actual children.

Gardner's *Annotated Alice* presents a convincing case (a "nice knock-down argument", as Humpty Dumpty remarks to Alice — "glory", he calls it, being the master of the words he uses: Gardner glosses this classic Dumptyism by outlining the Medieval theory of Nominalism, considering Aristotelian syllogisms, and the tension between propaganda and poetry in manipulating or playing with the meanings of everyday words) that Carroll's lines NEED to be read between. We can read the "Alice" books as a lightly humorous children's dream-fantasy: sometimes eerie, like a dream, sometimes amusingly puzzling. Or we can read the books as curious adults, and we will find more.

As Gardner says in his Introduction:

"The fact is that Carroll's nonsense [as the word is used when discussing Edward Lear, for example: non-realistic fantasy] is not nearly as random and pointless as it seems to a modern American child who tries to read the ALICE books. ... Children today are bewildered and sometimes frightened by the nightmarish atmosphere of Alice's dreams ... It is only to such adults [scientists and mathematicians and others who relish the ALICE books] that the notes of this volume are addressed."

(In passing, let me rule out adult readers finding any vestige of unnatural interest in little girls. There is no evidence to suggest Carroll was anything other than innocent, naïve and decent in his relationships with the young girls who interested him, however strange he may seem by modern standards. Similarly, with the hippy-era sneeringly smirking sly dig at the caterpillar smoking who-knows-what in a bizarre pipe, sitting on a possibly hallucinogenic mushroom, as a way of explaining Carroll's unusual fantasy — again, there is absolutely no evidence of drug-taking in Carroll's life. These suggestions are mere slanderous innuendo. The mud is in the mind of the mud-slinger. Gardner is very clear on these points, also.)

In my opinion it is helpful, and rewarding, to grasp the subtle wit and references throughout the "Alice" books and the "Snark". Many readers in Carroll's time would have been familiar with some of these passing hints and allusions. But with passing time, and changing culture, modern readers may not be immediately familiar. Moreover Carroll was theologically and philosophically trained, and a creative mathematician: he naturally includes other allusions that only theologically or philosophically trained readers, or academic mathematicians, would easily recognize. The annotations really throw light on Carroll's richness.

It takes a moment to compare Gardner's page-layout style of annotation with, for example, Donald J. Gray's similar annotated edition of Carroll's "Alice" books and other Carrollania and academic essays on Carroll (1971). In Gardner's *Annotated Alice* the original text by Carroll occupies a large font-size column, while Gardner's numbered annotations appear in smaller font-size in an adjacent margin-column. By contrast, Gray uses footnotes — and consequently often has much less space to explain important details.

Interestingly, the American publisher Norton has recently adopted Gardner's method, and two-column layout for annotations, to produce a fascinating sequence of annotated editions of such classics as *The Wind in the Willows, The Secret Garden, A Christmas Carol, Huckleberry Finn, Dracula,* and *Sherlock Holmes.* All highly recommended, although some of the American annotators miss the British subtleties of Dickens or Kenneth Grahame.

The Pick of the Gardner Crop?

I am sure different people will find different things that especially appeal in Gardner's diverse output. If I am pressed to play favourites (and when I am ever asked for my Top Ten All-Time Favourites in mathematics activities, Gardner is always at the top!) I would suggest:

— Pentominoes: but that leads on, immediately to polyominoes, generally, polycubes, Blokus games, Cathedral, space-filling, tessellation, square-coloring, handedness, rotations and reflections and symmetry,...;

— the Soma cube: this leads, similarly, to polycubes, cube-coloring, board games, space-filling, ...;

— Dominoes;

— Phi;

— Strategy games of many kinds, but especially those where the nature of the playing piece indicates how the piece moves;

— Scientific induction games, such as Eleusis, and Patterns;

And so on ...

I will not explain any of these, as I am sure they are easy to find on the internet. But perhaps that comment leads to the reason for writing this sketch of what I like about Martin Gardner. One increasing difficulty with the internet is its sheer size. How can we find good things in the internet, except by luck, or because someone tipped us off. This is my tip — Gardner is terrific. Google will show you how and why! But you really ought to have the books themselves!

References and Further Reading

Blokus: invented by Bernard Tavitian http://www.blokus.com [last accessed 16 August 2006].

Cathedral: The Game of the Medieval City; invented by Robert P. Moore 1981; published by Milton Bradley, New York.

Gardner, M. (1959). *Mathematical Puzzles and Diversions.* Simon & Schuster, New York: Harmondsworth: Penguin, 1965.

Gardner, M. (1960). *The Annotated Alice.* Clarkson N. Potter, USA: Penguin, Harmondsworth, 1965: and later revised editions.

Gardner, M. (1960). 'Mathematics Games' column, *Scientific American,* September 1960.

Gardner, M. (1961). *More Mathematical Puzzles and Diversions.* Harmondsworth: Penguin.

Gardner, M. (1967). *The Ambidextrous Universe.* Penguin, London (and a revised edition, later).

Gardner, M. (1973). "Mathematical Games: Racetrack", *Scientific American,* January, pp. 108–111.

Gardner, M. (1977). *Mathematical Magic Show.* Harmondsworth: Penguin.

Gardner, M. (1979). *Mathematical Circus.* Harmondsworth: Penguin.

Gardner, M. (1986). *Knotted Doughnuts and Other Mathematical Entertainments.* Freeman, New York.

Gardner, M. (1996). *The Night is Large: Collected Essays 1938-1995.* St Martin's Press, New York: Penguin, London.

Golomb S.W. (1954). "Checker Boards and Polyominoes". *American Mathematical Monthly.*

Gough, J. (2000). *Game, Set, and Match — Maths!* Australian Association of Mathematics Teachers [AAMT], Adelaide.

Gough, J. (2001). *Learning to play: Playing to learn — Mathematics Games That Really Teach Mathematics.* Mathematical Association of Victoria [MAV], Brunswick.

Gough, J. (2006). "Editorial: Do You Know Blokus?", *Vinculum,* vol. 43, no. 4, p. 2.

Gough, J. (2010). *Brain-Boosting Mathematics Games: Are You Game?* AAMT [Australian Association of Mathematics Teachers], Adelaide.

Gray, D.J. (Ed.) (1971). *Alice in Wonderland and Lewis Carroll* [Norton Critical Edition] Norton, New York: second edition 1992.

Hill, T., & Gough, J. (1992). *Work It Out With Maths Games.* Oxford University Press, Melbourne.

Lovitt, C., & Clarke, D. (1988). *The Mathematics and Curriculum and Teaching Program (MCTP) Vol. 2* Curriculum Development Centre; Canberra, "Four Cube Houses" pp. 505-510.

O'Beirne, T. H. (1961). "Pentominoes and Hexiamonds", *New Scientist* vol. 12, pp. 379-380.

Sackson, S. (1969). *A Gamut of Games.* New York: Random House.

Sackson, S. (1991). *The Book of Classic Board Games,* Klutz Press, Palo Alto.

Wells, D. (1997). *Penguin Book of Curious and Interesting Mathematics.* Penguin, London.

PLAYING WITH *WOLFRAM|ALPHA*

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Wolfram | Alpha is a web based 'computational knowledge engine' which incorporates Mathematica's computational capabilities and more. It is a bit like a mathematically/scientifically oriented Google, although the basis for inquiry is data rather than articles relating to a topic of interest. Inquiries can be made informally as long as they are basically well formed. This paper explores some possibilities for using Wolfram | Alpha.

What is a computational knowledge engine?

Roughly speaking a computational knowledge engine is a web-based resource that seeks to respond to queries in terms of available data rather than provide linking references to articles, texts or other sources that may contain related information (HREF1). *Wolfram | Alpha* (HREF2) was launched by Wolfram Research in early 2009, and can be considered as a complementary resource to search engines such as Google. A computational knowledge engine seeks to answer various classes of queries directly, rather than identify documents that refer to the subject of the query, thus, *Wolfram | Alpha* also provides access to computational capabilities of *Mathematica*. There is already a composite search engine *Gooftam* (HREF3) which submits the same query to both *Google* and *Wolfram | Alpha* and displays the results in a vertical split-screen arrangement, for example, see what entries such as 'number' or 'geometry' return. As with any web-based resource, *Wolfram | Alpha* has it's strengths and limitations, for example, it cannot be used to organize a holiday and identify bargain price options. On the other hand it can provide access to up-to date data on matters such as the frequency and strength of earthquakes over a given period of time.

Some general queries

As with many things, an empirical approach is informative. *Wolfram | Alpha* works with a single line query entry as shown in Figure 1.

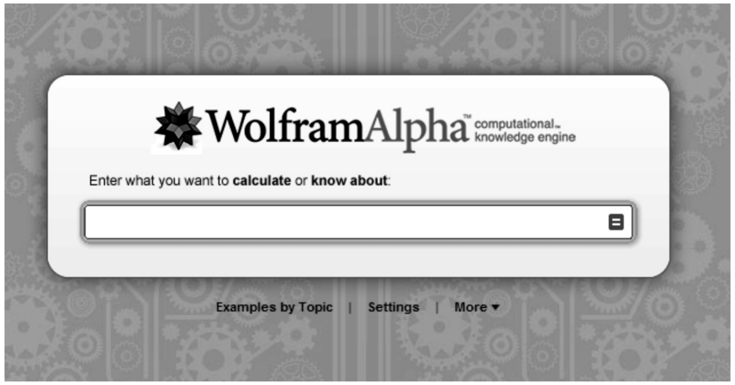

Figure 1: *Wolfram* | *Alpha* query line entry

The following are some suggested queries:

- the name of a city such as *Melbourne*
- the first name of a person, such as *Anne*
- a word such as *earthquake*
- one of the suggested *Examples by Topic* such as *Health and Medicine*

Queries can be refined, for example one could ask for earthquakes in the past 48 hours, the past few days, or the past two weeks, as shown in Figure 2.

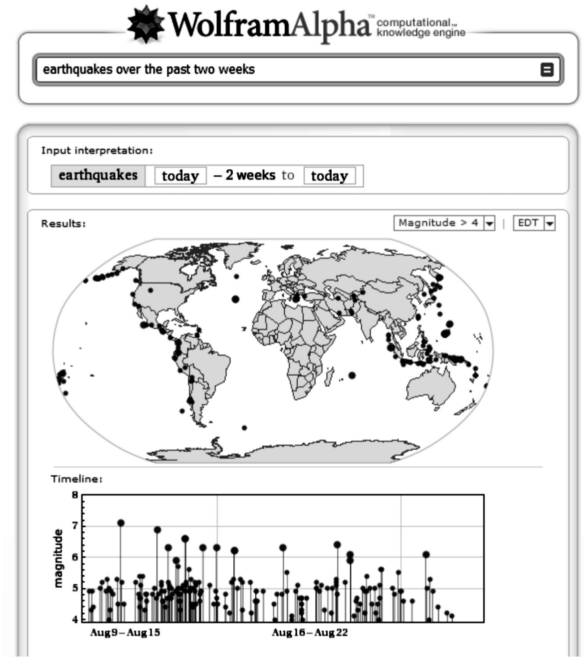

Figure 2: *Wolfram* | *Alpha* response to earthquake query

These queries could be used to support cross-curriculum connections of mathematics in, for example, Geography. As one builds up data from over increasing time periods (eg from 24 hours to several weeks or months) some interesting observations might be made with respect to the global 'graph' (such as plate tectonics and boundaries) and also the graph of the distribution of earthquakes and their intensity over time (HREF4).

Some mathematical queries

Wolfram | *Alpha* deals readily with queries of mathematical nature, for example, consider the familiar functions $f: R \to R, f(x) = x^2$ and $g: R \to R, g(x) = 2^x$. What does the graph of $f(x) \times g(x) = x^2 \times 2^x$ look like? Some preliminary 'in the head' and 'hand-waving' analysis indicates that the graph of this function will:

- be positive for all values in its natural domain except for at $x = 0$.
- be large and positive when x is large and positive

- be small and positive when x is large and negative
- pass through the points (0, 0); (1, 2), (2, 16); (-1, 2) and (-2, -1)

One would anticipate that a rough sketch of the graph displays asymptotic behaviour to y = 0 (from above) as x becomes increasingly large and negative, a local minimum at (0, 0) and a local maximum in the vicinity of x = -2. What does *Wolfram* | *Alpha* show? A naïve query such as in Figure 3 produces:

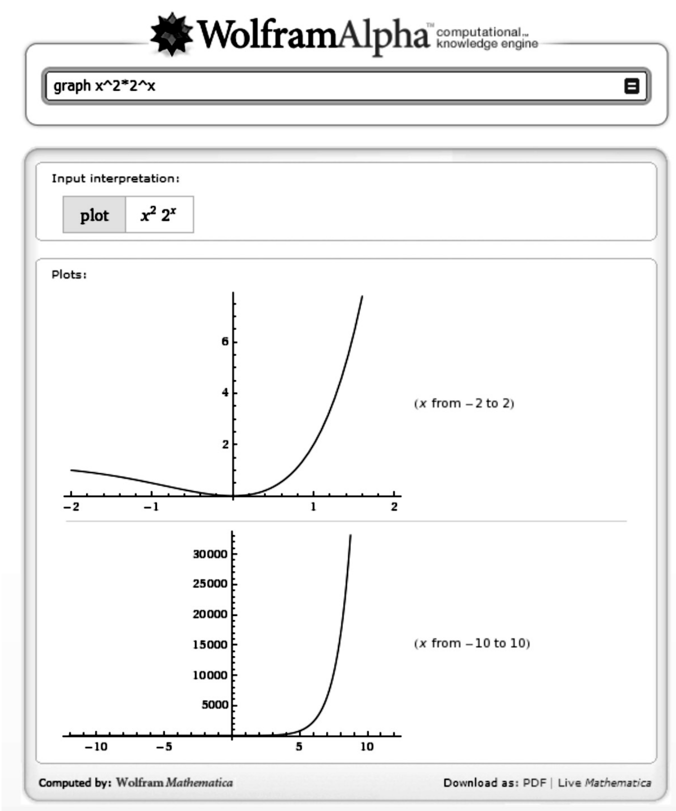

Figure 3: *Wolfram* | *Alpha* graph of $x^2 \times 2^x$

This can then be refined in terms of the graphing window to display the behaviour of the graph more fully, as shown in Figure 4.

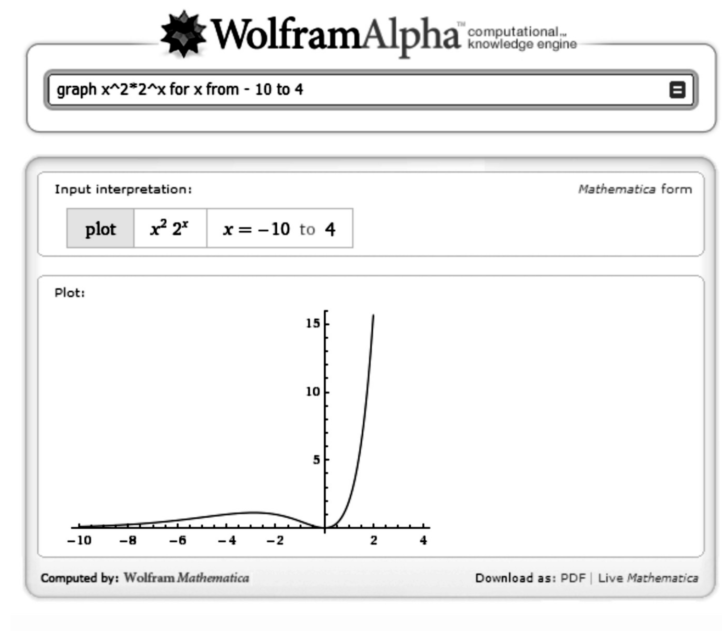

Figure 4: *Wolfram* | *Alpha* graph of $x^2 \times 2^x$ for x from -10 to 4

For further analysis one might differentiate the function and find the corresponding zeros to determine the local maximum. An interesting extension is to consider the set of values of $k \in R$ for which the graph of y = k intersects the graph of y = f(x) × g(x) at zero, one, two or three points. Similar and distinctive aspects of analysis can be carried out for the sum of two functions such as y =cos(x) + cos(3x), for example finding the solutions of cos(x) + cos(3x) = k for k = ½ , 0 or 2; or for a composite function such as $y = e^{-x^2}$, for example, location of the points of inflection. This composite function is also of interest leading into consideration of the standard normal distribution. In particular, one can investigate the area between the graph of this function and the horizontal axis as shown in Figure 5:

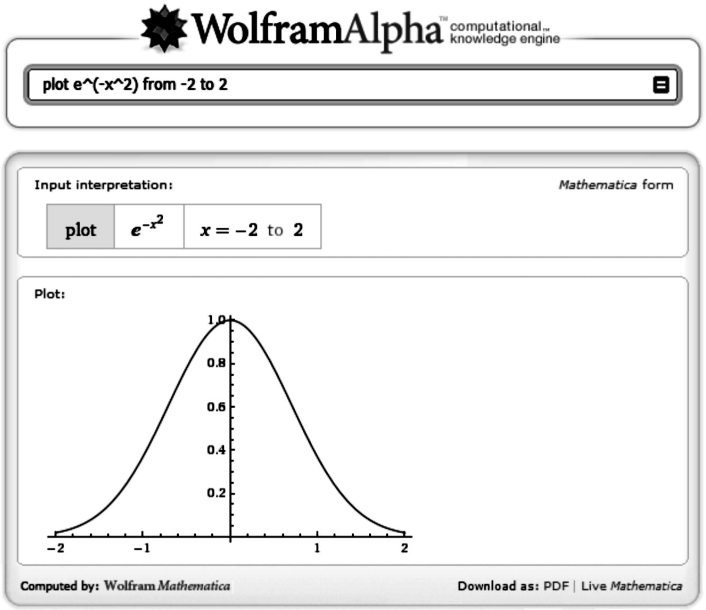

Figure 5: *Wolfram* | *Alpha* graph of the composite function

This graph can be used to make a quick estimate of a reasonable upper bound for the enclosed area of ½ × 4 × 1 = 2. The exact value of $\sqrt{\pi}$ is given in Figure 6.

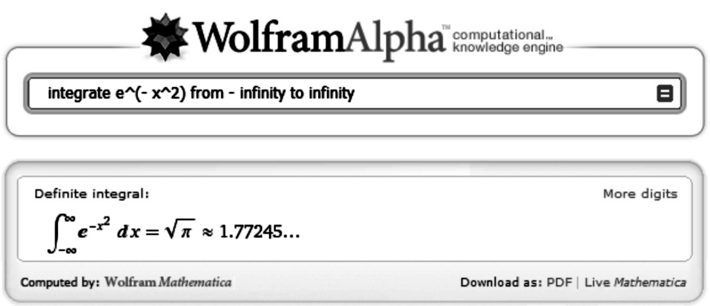

Figure 6: *Wolfram* | *Alpha* evaluation of area between curve and horizontal axis

From here one could proceed to consider the graphs of the probability density functions for the standard normal distribution and other normal distributions in terms of

the defining parameters, translations and dilations.

There are many other possibilities, for example exploring different probability distributions and their applications to modelling various phenomena.

Web Sites

HREF1: *http://en.wikipedia.org/wiki/Wolfram_Alpha*. Accessed 25/08/2010.
HREF2: *http://www.wolframalpha.com/*. Accessed 25/08/2010.
HREF3: *http://www.goofram... *

MATHEMATICAL GAMES: JUST TRIVIAL PURSUITS?

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In this paper the role of games in the teaching of mathematics is examined. A variety of games will be discussed to highlight key aspects of the use of games in the classroom. Theoretical and practical issues will be discussed in relation to the use of games in general and a selection of specific games.

Introduction

Authors such as Booker (2000) and Bragg (2006) have raised issues as to the effective use of games in the teaching of mathematics. Their work caused the authors (Swan & Hurrell) to examine the use of games in the teaching of mathematics more closely. After a brief review of the literature, some games will be presented and then discussed in order to highlight how the time spent playing mathematics games may be optimized. Suggestions will be made for ways of altering the games to meet the needs of a diverse range of learners.

Literature Review

Prior to beginning a discussion on the use of games it should be stated that the term 'games' has been used in a variety of ways. Gough (1999) defined a mathematical game as having mathematical content and needing to have two or more players who take turns

competing to achieve a 'winning' situation of some kind, each able to exercise some choice about how to move through the playing (Gough 1999, p. 12). Gough describes some 'games' as luck, races or dressed up worksheets. Other authors (Harvey & Bright, 1985; Oldfield, 1991) have also listed criteria to describe the characteristics of mathematics games.

The authors acknowledge that some of the games described in this article do not meet all the criteria listed by Gough, (1999), Harvey and Bright, (1985), Oldfield, (1991). The choice of games mentioned in this article was based on the potential for some meaningful mathematics to be developed as a result of playing the game.

Other important criteria includes:

- the opportunity to discuss the mathematics inherent in the game;
- simplicity of rules;
- ease of set up;
- short playing time; and
- the ability to make simple variations to cater for differing abilities.

Sullivan (1993, p. 211) included the characteristic "they do not rely on winners and losers for their interest" when listing his criteria for choosing certain games for learning mathematics. Criteria were developed in part, based on work with teachers where teachers' use of games was examined (Marshall & Swan, 2009; Swan & Marshall, 2009a; Swan & Marshall, 2009 b). It may appear trivial but criteria such as ease of set up and short playing time impact on which games are chosen by teachers and whether games are used at all.

The use of games in the teaching of mathematics has been presented in a positive light (Gough, 1999b; Ainly, 1990). However, poor game choice, over-using a particular game or using a game when the students are not ready for it will prove counterproductive. What is clearly important is the structure of the games used. Bragg (2006) who studied the use of two games in the teaching of mathematics, raised questions about the efficacy of explicit instruction and the use of games, along with issues related to grouping children according to ability in their game play. The literature does highlight, however, that if this structure is not provided learning does not always take place (Onslow, 1990; Burnett, 1992).

Games may be used for a variety of purposes, not simply as a means of practising previous taught skills. They also have a place as an instructional tool (Booker, 2000). In this article a variety of different games are provided, not just those designed to practise a particular skill. For a broader discussion on the use of instructional games and mathematics learning see Booker, Bond, Sparrow and Swan (2010) pp 28 - 29.

Games

In this section a variety of games are presented. The reader is encouraged to consider several of these games in the light of the criteria listed above and add any of their own. A variety of game types have been provided in order to evoke a variety of responses and to encourage readers to examine whether the criteria apply to all games or whether slightly different criteria may be applied according to the game type or purpose of the game.

Game 1 - Before and After

Materials: A pack of playing cards with the picture cards removed.

Aim: To place a card that either comes before or after another card.

This is a simple card game for 2 – 4 players. Five cards are dealt to each player and players take turns to place a card on the table. After the first player places a card, other players take turns to place a card that comes either one before or one after the card currently in play. In essence the each player in turn match a card that is one more or less than the card currently in play (the face up card). Players monitor that the correct card is being placed. The first player to dispose of his or her cards is the winner (Booker et al., 2010, p. 99).

Game 2 - Snap +/– 1

Materials: A pack of playing cards with the picture cards removed.

Aim: To be the first player to recognise a difference of one when two cards are laid on the table.

This game is played along similar lines to the standard game of Snap that involves being the first player to slap a deck of cards when a card that matches the one uppermost on the deck is played. In this case instead of a match the deck is slapped when there is a difference of one between the card uppermost on the deck and the one played.

Variation: This game may be varied by changing the rule from a difference of one to a total of ten (Swan, 1998, p. 24).

Game 3 - Splitter, Acey Deucy (In Between)

This game is based on a gambling game. The authors suggest referring to it as the 'In Between' game or 'Splitter'.

Materials: A pack of playing cards with the picture cards removed.

Aim: To decide the likelihood of a card lying between two other cards.

A game for two players where each player turns over two cards.

The player scores the difference between the two cards. For example if a 6 and 3 are turned over a score of 3 is recorded. If the cards are the same then the player does not score. The player may choose to turn over a third card. If the card falls numerically between the first two cards, then the player receives a bonus ten points. However, if the card does not fall numerically between the two cards then all points for that round are forfeited. Players must play the third card rather than forfeit a turn.

Play continues until one player reaches a set number of points, eg 50 or 100.

Variations: Gough described a version of this game using dice. Consider the implications for this game using two six sided dice or two ten sided dice. (Gough, 1993, p. 220)

These three games are similar in that they all involve cards, the conventions of card playing and mathematical content. They vary slightly in terms of the mathematical content while at the same time focus on some very important early number concepts. Clearly these games would be ideally suited to the needs of students in the early years and are somewhat out of place with older students. This helps to illustrate that while these games are essentially sound and meet most of the criteria listed earlier it would be a waste of time to spend time playing the games with older students, there would be little or no challenge and the students would become bored.

Trading games

A trading game helps students understand the place value system as trading or exchanging of ten ones to make one ten, or ten tens to make one hundred is the basis of the game. Generally players use a board split into three columns indicating ones, ten and hundreds. Dice are rolled and pop-sticks or objects are placed in the ones column until nine pop-sticks have been placed. As soon as the tenth pop-stick is placed in the ones column a trade or exchange has to take place, that is ten ones are traded for one ten. Trading games may be played in other bases, such as base four, in which case the trading rule changes. In the case of base four a trade would take place when four pop-sticks were shown on the playing board. Note a different board is required as the ones, tens and hundreds headings would no longer apply. A sequence of trading games is given in Booker et al (2010). Trading games are used to build understanding of the number system and in particular the concept of place value that underpins that system. (Booker et al, 2010 pp. 104 – 117; Swan & White, 2006).

Trading games may progress from simple bases, through to bundling pop-sticks on a place value mat and then to using Base Ten Blocks (MAB).

Trading games in Base Ten have been examined in detail. See Understanding place value: a case study of the Base Ten Game (DEST, 2004).

In addition to examining the value of this game in developing an understanding of place value, the researchers also examined how the teachers altered the game and what complementary activities teachers used to make the concepts inherent in the game explicit. The results of this research help to illustrate that games are best used in concert with other well chosen tasks and activities that support the development of the associated mathematics concept.

Linking to the Australian Curriculum

At the time of writing, access to the Australian Curriculum was limited to the draft version. The authors have used examples from the draft curriculum to illustrate how direct links may be made between specific mathematics content to be taught and certain games. The following links are cited as examples of how the game Battleships may be linked to the Australian Curriculum (ACARA, 2010).

Year 5: Measurement and Geometry – Location

Describe locations and routes using a coordinate system such as road maps, the four main compass directions and the language of direction and distance.

Year 6: Measurement and Geometry – Location

Describe and interpret locations and give and follow directions, using scales, legends, compass points, including directions such as NE and SW, distances, and grid references.
In Year 7 students are taught to plot points on the Cartesian plane using all four quadrants.

Game 4 - Battleships Game

Materials: Attacking and defending grids.

Aim: To sink all of your opponent's ships.

A game for two players. Both players place their ships on their own "defending" grid by placing the appropriate letters -AAAAA for an aircraft carrier, BBBB for a Battleship, CCC for a Cruiser and DD and DD for two destroyers - horizontally, vertically or diagonally. All of the letters are to be in a straight line and adjacent to each other.

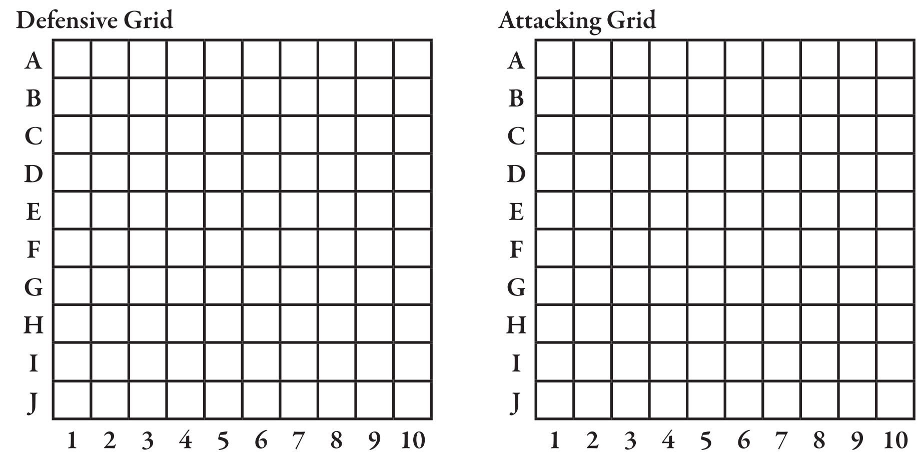

Figure 1: Battleship game grids

Whoever goes first calls out a position (i.e. G-6). The other player says either "Hit" or "Miss" depending upon whether one of his ships is in the position called out. The person calling marks a hit or a miss on the "attacking grid" to keep track of the shots. The other person marks the shot on the "defending grid". If the shot is a "Hit", the player goes again – otherwise the 2nd player takes a turn. Once one player has scored a hit on all of the spaces for a particular ship, the opponent must call out "Hit...you have sunk my Cruiser" (or whatever type of ship it was).

Clearly in playing this game students need to use co-ordinates, interpret locations and locate points on the Cartesian plane. These are all elements found in the Geometry and Measurement Strand: Location, found in the draft Australian mathematics Curriculum (ACARA, 2010).

Monitoring the mathematics

When students are actively engaged in playing mathematics games teachers have the opportunity to observe them playing. If students are encouraged to verbalise their moves then a teacher may notice what moves are being made and hear what is being said. Asplin (2003) noted in her study of the card game Numero, that much better gains in terms of student understanding were made by those students who verbalized the moves they were making. As students discuss moves and weigh up strategies, opportunities to teach will arise. Teachers need to be alert to these opportunities when they occur. However, a teacher cannot attend to all groups of students when they play games therefore alternative strategies need to be put in place for monitoring progress.

Consider the role of a 'banker' in trading games and how exchanges may be monitored by that player. The role may be rotated among the players so that all players have the opportunity to take on this role. Some player may require extra support in terms of a ready reckoner or a table that may be used to check various moves. The board games 'Get In Shape' and 'Treasure Trove' (Swan, 2010) involve one player taking on the role of a 'gym instructor' (checker) or Captain (checker). While the game is in progress this player monitors all the moves made by each of the other players. After the game, students may be offered similar experiences to assess the learning that has taken place. For example, the game 'Treasure Trove' involves recognising various multiples. Students' understanding of these multiples may be assessed by asking them to colour in a number track to show a particular multiple.

Conclusion

Every teacher has a grab bag of favourite games. The authors hope that a few games may now be added to that collection. A quick review of past MAV conference proceedings and the internet will provide even more games that may be added. Regardless of the number and type of games, the authors encourage readers to re-examine these maths games in the light of what has been discussed.

Consider:

- What is the mathematics inherent in the game?
- Does the mathematics need to be made explicit? If so how?
- Do the students possess the pre-requisite skills required to play the game?
- Is the mathematics inherent in the game appropriate for the current needs of the students? Can the game be adapted to fit the needs of less able or more able students?

After considering the points above it is possible that a favourite game might be packed away for the time being. This may not mean that the game itself is flawed but simply that it does not meet the needs of the students at that particular time.

References

ACARA. (2010). *Australian Curriculum: Mathematics (Draft).* Retrieved from http://www.acara.edu.au/curriculum.html

Ainly, J. (1988). Playing games and real mathematics. in D. Pimm (Ed). *Mathematics teachers and children; a reader* (pp. 239 – 248) London: Hodder and Stoughton in association with the Open University.

Asplin, P. (2003). *An investigation into the effectiveness of Numero in learning mental mathematics in Year 6.* Unpublished honours thesis, Curtin University, Perth, Australia.

Bragg, L., (2006). *The impact of mathematical games on learning, attitudes and behaviours [unpublished doctoral thesis].* La Trobe University, Bundoora.

Booker, G. (2000). *The maths game: using instructional games to teach mathematics.* Wellington: NZCER.

Booker, G., Bond, D., Sparrow, L., & Swan, P. (2010). *Teaching primary mathematics 4th Edition.* NSW: Pearson.

Burnett, L. (1992). *Using instructional games to construct number understanding and skills.* in M. Horne & M. Supple (Eds.), *Mathematics: Meeting the challenge* (pp. 223-228). Melbourne: The Mathematical Association of Victoria.

Gough, J. (1993). Playing games to learn maths . In J. Mousley & M. Rice (Eds.), *Mathematics of primary importance* (pp. 218-221). Brunswick, Victoria: The Mathematical Association of Victoria.

Gough, J. (1999). *Playing mathematical games: When is a game not a game?* Australian Primary Mathematics Classroom 4(2), 12 – 15.

Harvey, J. G., & Bright, G. W. (1985). *Basic math games.* Palo Alto, California: Dale Seymour Publications.

Marshall, L., Swan, P. (2009). *Games: A catalyst for learning or busy work?* Proceedings of the Conference of the Australian Association of Mathematics Teachers: Fremantle.

Oldfield, B. J. (1991a). *Games in the learning of mathematics - Part 1: Classification. Mathematics in School,* 20(1), 41-43.

Oldfield, B. J. (1991b). *Games in the learning of mathematics - Part 2: Games to stimulate mathematical discussion. Mathematics in School,* 20(2), 7-9.

Onslow, B. (1990). *Overcoming conceptual obstacles: The qualified use of a game. School Science and Mathematics,* 90(7), 581-592.

Sullivan, P. (1993). *Short flexible mathematics games.* in J. Mousley & M. Rice (Eds.), Mathematics of primary importance (pp. 211-217). Brunswick, Victoria: The Mathematical Association of Victoria.

Swan, P. (1998). *Card Capers*. Perth: A-Z Type.

Swan, P. (2010). *Treasure Trove:* Perth: Abacus Educational.

Swan, P. (2010). *Get in Shape.* Perth: Abacus Educational

Swan, P., & Marshall, L. (2009a). *Mathematics games: Time wasters or time well spent?* Proceedings of the Maths into the 21st Century Conference: Dresden.

Swan, P., & Marshall, L. (2009b). *Mathematics games as a pedagogical tool.* Proceedings of the International Conference on Science and Mathematics Education. Penang: CoSMEd.

Swan, P., & White, G. (2006). *Developing maths with Base Ten. Perth: RIC Publications.*

Understanding place value: a case study of the Base Ten Game. (2004). Retrieved from: (http://www.dest.gov.au/sectors/school_education/publications_resources/profiles/case_study_the_base_ten_game.htm)

SOME RATIONAL NUMBER COMPUTATIONS

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The rational numbers are a well known and important part of the school curriculum often referred to in terms of 'fractions' and 'decimals' and 'ratios, proportions, percentages and rates'. This paper considers some other aspects of rational numbers such as ' how many rational numbers are there?'; 'which fractions have finite decimal expansions?'; 'how can one place any two fractions on the same number line exactly?'; 'what are Egyptian fractions?'.

The rational numbers

The rational numbers can be defined as the set of numbers of the form $\frac{m}{n}$ where m and n are integers and n is non-zero, with the usual arithmetic operations and order relation. The expression $\frac{m}{n}$ is a *quotient*, commonly called a *fraction*, and the letter Q is used to denote the set of rational numbers. If $\frac{m}{n}$ is evaluated as $m \div n$ then the corresponding decimal expansion is obtained. This may be finite as in $\frac{3}{4} = 0.75$, or *infinite recurring* as in $\frac{2}{3} = 0.666... = 0.\overline{6}$. The term *rational* number is used as $\frac{m}{n}$ can also be though of as expressing a *ratio* of integers $m : n$ where m and n are referred to as the *numerator* and *denominator* of the fraction respectively. The characteristic 'horizontal line' which separates the numerator and denominator is called a *vinculum* (Latin for 'tie' or 'bond', first used in mathematics much as we do by the Arabs, then by Fibonacci in Europe in the 13th Century CE). The diagonal slash alternative, called a *virgulus*, was subsequently introduced for (single-line) typesetting convenience – and is still used in this way by many teachers in their worksheets today! The rational numbers could be just as well represented as ordered pairs of the form (*numerator*, *denominator*) with the arithmetic operations and order relation defined accordingly, for example (a, b) + (c, d) = (ad + bc, bd). This is essentially the approach used by the Hindus and others before the Arabs introduced the vinculum as a refinement of the Hindu form. Each rational number corresponds to an infinite set of equivalent fractions, for example A fraction is said to be in simplest form (simplest terms) if its numerator and denominator are co-prime, that is, they have no common factors greater than 1 (alternatively, their greatest common divisor is 1). If a fraction = (m, n) is in simplest form, then its family of equivalent fractions (equivalence class) is {(km, kn) where k Z\ {0}}. This equivalence forms the basis for the operations of addition and subtraction of fractions, and the ordering relation. Whatever approach to representation is taken, the rational numbers form a mathematical structure known as an ordered field (HREF1).

How many rational numbers are there?

Two sets have the same size (magnitude, cardinality) if some one-to-one correspondence can be established between them (this correspondence does not have to be order-preserving). The set of natural numbers N = {1, 2, 3 ...} is a countable and infinite set. Countable basically means one can say " here is the first element, the second element, the third element ... and so on". Infinite means that it can be put into a one-one correspondence with a proper subset of itself. N is the smallest infinite set. Q must be at least as big as N (consider the infinite sequence). It is also the case that Q is the same size as N. there are various ways of constructing the required one-to-one correspondence (see, for example HREF1). One approach is to consider the set of all points in the cartesian plane whose coordinates are of the form (m, n) where m and n are integers. This set certainly includes all rational numbers expressed as ordered pairs. Form a 'square spiral' that starts at the origin (0, 0) and takes unit length horizontal or vertical 'steps' to pass through each of the points of the specified form, for example

(0,0) → (1, 0) → (1,1) → (0, 1) → (-1, 1) → (-1, 0) → .... This will pass through every ordered pair, thus every rational number will be reached somewhere in the sequence of steps {first step, second step, third step .... }. Some tidying up can be applied. Any ordered pair with second element 0 can be deleted (it does not correspond to a rational number); also repetitions corresponding to equivalent fractions can be deleted, for example (6, 8) is not required once (3, 4) has been included. The first few terms of this listing are: . Thus there are as many rational numbers as natural numbers. For any rational number the members of its equivalence class correspond to those integer values ordered pairs, with the exception of (0, 0), that lie on the straight line through the origin defined by the relation ny = mx, or alternatively the solutions of the Diophantine equation mx $(a, b) + (c, d) = (ad + bc, bd)$.

This is essentially the approach used by the Hindus and others before the Arabs introduced the vinculum as a refinement of the Hindu form. Each rational number corresponds to an infinite set of equivalent fractions, for example $0.75 = \frac{3}{4} = \frac{6}{8} = \frac{9}{12} = \ldots$ A fraction $\frac{m}{n}$ is said to be in simplest form (simplest terms) if its numerator and denominator are co-prime, that is, they have no common factors greater than 1 (alternatively, their greatest common divisor is 1). If a fraction $\frac{m}{n}$ = (m, n) is in simplest form, then its family of equivalent fractions (equivalence class) is $\{(km, kn)$ where $k \in Z\backslash\{0\}\}$. This equivalence forms the basis for the operations of addition and subtraction of fractions, and the ordering relation. Whatever approach to representation is taken, the rational numbers form a mathematical structure known as an ordered field (HREF1).

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$mx - ny = 0$, as partially illustrated in Figure 1:

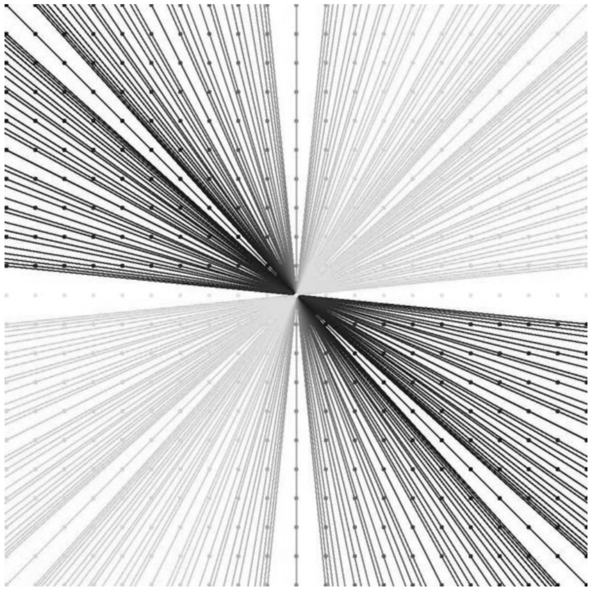

Figure 1: graphical representation of some equivalent fractions

Which fractions have finite decimal expansions?

It is possible to identify, *prior* to division, those fractions that have *terminating* decimal expansions by inspection of the denominator of the fraction. If this denominator is a product of non-negative integer powers of 2 and 5, that is, of the form $n = 2^r \times 5^s$ for some non-negative integers r and s, then the fraction will have a terminating decimal expansion. Table 1 shows the decimal expansions, correct to 50 significant figures of the fractions of the form $\frac{1}{n}$ for $n = 20$ to 40:

Table 1: decimal expansion of some unit fractions

$\frac{1}{20}$	0.050000000000000000000000000000000000000000000000000
$\frac{1}{21}$	0.047619047619047619047619047619047619047619047619048
$\frac{1}{22}$	0.045454545454545454545454545454545454545454545454545
$\frac{1}{23}$	0.043478260869565217391304347826086956521739130434783
$\frac{1}{24}$	0.041666666666666666666666666666666666666666666666667
$\frac{1}{25}$	0.040000000000000000000000000000000000000000000000000
$\frac{1}{26}$	0.038461538461538461538461538461538461538461538461538
$\frac{1}{27}$	0.037037037037037037037037037037037037037037037037037
$\frac{1}{28}$	0.035714285714285714285714285714285714285714285714286
$\frac{1}{29}$	0.034482758620689655172413793103448275862068965517241
$\frac{1}{30}$	0.033333333333333333333333333333333333333333333333333
$\frac{1}{31}$	0.032258064516129032258064516129032258064516129032258
$\frac{1}{32}$	0.031250000000000000000000000000000000000000000000000
$\frac{1}{33}$	0.030303030303030303030303030303030303030303030303030
$\frac{1}{34}$	0.029411764705882352941176470588235294117647058823529
$\frac{1}{35}$	0.028571428571428571428571428571428571428571428571429
$\frac{1}{36}$	0.027777777777777777777777777777777777777777777777778
$\frac{1}{37}$	0.027027027027027027027027027027027027027027027027027
$\frac{1}{38}$	0.026315789473684210526315789473684210526315789473684
$\frac{1}{39}$	0.025641025641025641025641025641025641025641025641026
$\frac{1}{40}$	0.025000000000000000000000000000000000000000000000000

In the case where the denominator is 40, this can be expressed as $40 = 8 \times 5 = 2^3 \times 5^1$ and the corresponding decimal fraction is terminating where $\frac{1}{40} = 0.025000\ldots = 0.025$. Why should this be the case? For a fraction $\frac{1}{n}$ to have a terminating, or finite, decimal expansion, it must have an equivalent fraction with denominator of the form 10^k for some natural number k. Since 2 and 5 are the only prime factors of 10, this means that n can only have powers of 2 and/or 5 as its factors. Clearly, if $\frac{1}{n}$ has a terminating decimal expansion, then so does $\frac{m}{n} = m \times \frac{1}{n}$. The converse is also the case - every number with a finite or infinite recurring decimal expansion can be written as a unique fraction (up to equivalence).

For example, $3.142 = 3 + \frac{1}{10} + \frac{4}{100} + \frac{2}{1000} = 3\frac{142}{1000}$ and, in general for a finite decimal expansion $a.b_1b_2b_3...b_n = a + \frac{b_1b_2b_3...b_n}{10^n}$, where $b_1b_2b_3...b_n$ is the sequence of digits to the right of the decimal place.

To express a number with an infinite recurring decimal expansion as a fraction, there are two main approaches, using the limiting sum of a convergent infinite geometric series, or by arithmetic operations based on multiplication by a suitable power of 10. For example, consider the rational number x with the infinite recurring decimal expansion: $x = 0.\overline{23} = 0.2323232323...$. Then$10_2x$ = $100x = 23.\overline{23} = 23.2323232323...$, so $100x - x = 99x = 23 \Rightarrow x = \frac{23}{99}$.

How can one place any two fractions on the same number line exactly?

Numbers are not points. However a point *corresponding* to any rational number $\frac{m}{n}$ can be located on a number-line by similarity, since it's position left or right of a selected origin (depending on whether it is negative or positive) is determined by the endpoint of a given unit segment being re-scaled to length $m \times \left(\frac{1}{n} \times 1\right)$. For example, the point corresponding to the fraction $\frac{2}{3}$ is constructed as shown in Figure 2. Let the line containing points corresponding to a particular choice of 0 and 1 and hence a given unit segment [0, 1] be the reference number line. Construct a line containing 0 and 0' perpendicular to the line containing 0 and 1. Now construct a line containing points 0', 1', 2' and 3' based on a different unit segment [0', 1'] parallel to the original line, such that the length of [0', 3'] is greater than the length of [0, 1]. The distance between 0 and 0' is to a certain extent arbitrary given that it is sufficient for convenient construction. Locate *P* by producing the line from 3' through 1 on the reference number line until it meets the common perpendicular to the pair of parallel lines. The triangle formed by the points *P*, 0 and 1 is similar to the triangle formed by the points *P*, 0' and 3'. Likewise, the triangle formed by the point *P* and points for 0 and ⅔ is similar to the triangle formed by the point *P*, and those for 0' and 2'.

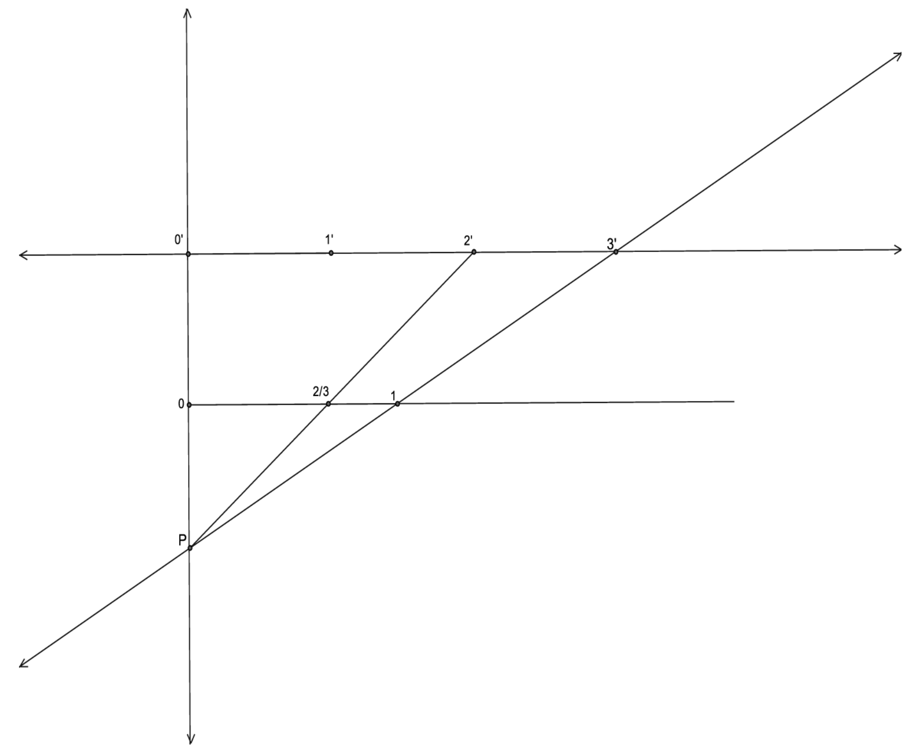

Figure 2: construction of the point corresponding to ⅔

That is, the 2:3 ratio of lengths of the segments [0', 2'] and [0', 3'] on the line containing points 0', 1', 2' and 3' corresponds to the ⅔: 1 ratio of lengths of the segments [0, ⅔] and [0, 1] on the line containing 0, ⅔ and 1.

To place another fraction, for example $\frac{3}{5}$, on the *same* number line as 0, 1 and ⅔ a new parallel line containing points 0''', 1''' ... 5''' and the corresponding similar triangles would need to be constructed. The common vertex for these similar triangles would also lie on the perpendicular but be distinct from *P*. Dynamic geometry tools, as well as compass and straight edge, are well suited to exploring this type of construction.

Egyptian fractions

The ancient Egyptians based their fraction arithmetic on the use of positive unit fractions, that is, fractions of the form $\frac{1}{n}$ where $n > 0$ (HREF2; HREF3; HREF4). Addition and multiplication of unit fractions, and ordering of fractions is relatively simple,

$\frac{1}{a}+\frac{1}{b}=\frac{a+b}{ab}; \frac{1}{a}\times\frac{1}{b}=\frac{1}{ab}; \frac{1}{a}<\frac{1}{b}\Leftrightarrow a>b$ for $a, b > 0$. As an Egyptian fraction, a positive fraction $\frac{m}{n}$ where $m < n$ would be written as a sum of unit fractions, without repetition of the denominator, for example: $\frac{3}{4}=\frac{1}{2}+\frac{1}{4}$; $\frac{1}{3}+\frac{1}{5}+\frac{1}{7}=\frac{71}{105}$. Any non-unit fraction between 0 and 1 can be written as an 'Egyptian fraction' using the following algorithm – "Start with the given non-unit fraction and keep on subtracting a suitably large but not too large unit fraction, which has not yet been used, from the previous result until a unit fraction answer results. Then form the required sum by backtracking". For example given $\frac{m}{n}=\frac{19}{35}\rightarrow\frac{19}{35}-\frac{1}{2}=\frac{3}{70}\rightarrow\frac{3}{70}-\frac{1}{25}=\frac{1}{350}$ so $\frac{19}{35}=\frac{1}{2}+\frac{3}{70}=\frac{1}{2}+\frac{1}{25}+\frac{1}{350}$. Is this representation unique? No! If $\frac{1}{24}$ had been used instead of $\frac{1}{25}$ then $\frac{19}{35}=\frac{1}{2}+\frac{1}{24}+\frac{1}{840}$ is obtained. If $\frac{1}{30}$ had been used instead of $\frac{1}{25}$ then $\frac{19}{35}=\frac{1}{2}+\frac{1}{30}+\frac{1}{105}$ is obtained. There are many interesting investigations involving Egyptian fractions: Is there an analogue of a 'simplest' form for Egyptian fractions?, Can Egyptian fraction representations be 'extended indefinitely'?, How can Egyptian fractions be used to compare the size of non-unit fractions or to share quantities?

Web Sites

HREF1: *http://en.wikipedia.org/wiki/Rational_number*. Accessed 26/08/2010.

HREF2: *http://en.wikipedia.org/wiki/Egyptian_fractions*. Accessed 26/08/2010

HREF3: *http://www.maths.surrey.ac.uk/hosted-sites/R.Knott/Fractions/egyptian.html* Accessed 26/08/2010.

HREF4: *http://mathworld.wolfram.com/EgyptianFraction.html*. Accessed 26/08/2010

AUSTRALIAN MATHEMATICS CURRICULUM IMPLEMENTATION: TEACHER REFLECTIONS ON STUDENT THINKING INFORMING FUTURE DIRECTIONS

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How can we help to develop the deep understandings advocated in the Shaping Paper for the Australian Mathematics Curriculum? This article shows deep understandings may not be present even though a topic has been taught in previous years or earlier in the year. It describes a teaching and research approach that provided a 'window' into what students knew, and reports teacher reflections that helped refine an exploratory task to increase its learning potential. In particular, this paper shows that we need to increase the variation in images students are exposed to when learning Geometry, and increase opportunities for students to generate and analyse such images for themselves.

Introduction

My research over the past ten years has focused on what helps students to creatively develop new mathematical knowledge. My interest developed as a secondary teacher searching for ways to engage students in mathematics learning. I was initially surprised when they reported that they not only enjoyed mathematics more but also understood more than when taught by conventional 'chalk and talk' methods. My research (Williams, 2005) (and international research) confirms my findings as a teacher: when students are 'told' rather than able to explore they tend to develop a fragmented set of rules and procedures that do not represent what the teacher intended. Eden (pseudonym), a student in a Year 8 class in my earlier research explained that it was important to work thing out himself:

"Ohh, you try to work everything out for yourself because, that way, you know everything (pause) sort of (pause) you will be able to think clearer for tests and whatever."

In other words, where tests required use of mathematics in unfamiliar ways, Eden knew he needed to develop his own understandings to be able to select and use appropriate mathematics. This article describes the contexts for my present study, and illustrates student responses (including some who performed at a high-level on conventional class tests) showing various limited initial understandings that surprised their teachers. Teachers' subsequent reflections helped me to re-design the task. The initial task is presented then later the refined task.

The Context

This paper draws on research in elementary classrooms (students in Grades 3-6), and some findings from Year 8 classrooms. Schools in these studies were from government and Catholic School sectors, from three metropolitan regions (Northern, Southern, and Outer Eastern). Cultural mixes in schools differed, as did proportions of students new to Australia. Even so, findings reported were common across classes, levels, schools, sectors, and students performing at different levels on traditional mathematics tests. They fit with international research about students learning rules and procedures rather than talking and exploring to make meaning.

Task: Shaping Rectangles

This task (Figure 1) addresses an area of mathematics identified by teachers as requiring improvement—multiplication tables and their uses. This task was previously designed to link rectangle dimensions, factors, and total number of squares required to fill a whole rectangle, as part of an informal development of mathematical ideas underpinning areas of rectangles. Students search for: the number of possible rectangles made using all of a given number of squares; patterns associated with the rectangles they find (e.g., multiplying dimensions of sides gives total number of squares in rectangle); and reasons why these

patterns occur (requires 'seeing' arrays). It should also consolidate multiplication.

Activity 1. In groups, work out all the things you know about rectangles ready to share your ideas with the rest of the class. Find the smallest number of things you would need to tell someone so the figure they draw has to be a rectangle.

Activity 2. Your group are going to think about the rectangles you can make with square tiles (top surface of tiles lying flat on table). Always give all rectangles that are possible for each question.

Each rectangle must use all the square tiles allowed for that question.

Insides of rectangles must be filled with square tiles with no empty spaces.

There are fourteen square tiles on your desk to help you think about the questions. You will not be given more tiles. Your group will give a 1-2 minute report to the class each 5-10 minutes.

Qn 1. What rectangles can your group make that each use all of 14 square tiles?

Qn 2. What rectangles can your group make that each use all of 12 square tiles?

Qn 3. What rectangles can your group make that each use all of 16 square tiles? You do not have enough square tiles to try this so you will need to think about it another way.

Qn 4. Can you always make more rectangles when you have more tiles? Why? Can you explain?

Qn 5. Can you give a mathematical argument (using whatever you like to help you: words, diagrams, numbers or whatever else helps) to show you have found all possible rectangles that can be made using 16 square tiles?

Qn 6. We are going to try to find the number of square tiles between 1 and 46 tiles that will make the most rectangles.

1. Search to find 'good numbers' (numbers that make many rectangles) and if possible the 'best number'.
2. Discuss the thinking you did to work this out (ready to report this to the class)
3. If you think you have found the 'best number', can you give a mathematical argument to show this? How can you show you have found all possible rectangles?

Implementing Task: Reporting sessions should occur about each 5-10 minutes. Put reports in an order so each group has a chance to contribute something new

Figure 1. Shaping Rectangles Task

Van Hiele's levels (Pegg & Currie, 1998) show a progression in what students might 'see' when they look at a figure. They might see the: shape only, shape and one feature, shape and two or more of features. In my study, some students saw only shapes and not features when they commenced Activity 1: "At first I only saw the shape (rectangle) not things about it". This activity was intended to raise student awareness of the features of rectangles.

Activity 2 was intended to support students as they find links between features of rectangles and 'see' and link arrays with other features. It increases student understanding of the nature of factors, develops and consolidates knowledge of tables through continual searching for possible factors, strengthens awareness of numbers that have numerous factors, and provides visual representations to support development of understanding of why certain tables have certain answers. I knew from my previous research that many Grade 5/6 students do not initially 'see' the arrays in rectangles when they are working in unfamiliar situations if they are not reminded they exist. It is important for students to 'see' and communicate the presence of these arrays (and how they help) because research has shown that such student-student discussions can build deep understandings. This paper focuses on student understandings of rectangles and right angles. My other paper in this book (Williams, in press) focuses on how students' understandings of links between rectangles, their dimensions, factors, arrays, and multiplication, developed as they worked with the Shaping Rectangles Task. The nature of angle in general is not addressed here.

Window Into Student Thinking

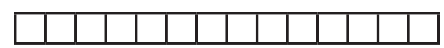
The Engaged to Learn teaching approach provided many opportunities for teachers to observe their students and gain understandings of what they knew (whilst I was the primary implementer of the task). This approach includes small group work, and short group reports to the class at intervals during the problem solving process. Group interactions and reporting sessions provided a 'window' into student thinking, and student reports give access to understandings a group has crystallized. For more information on a teacher stimulating group discussion about what to report, see Williams (2003).

Four video cameras capture each group (of 3-4 students) interacting, and student reports to the class as a whole. In post-lesson video stimulated interviews, students find on video (of own group and reporting sessions) the parts of the lesson they want to discuss, and talk about what they thought was happening, and what they were thinking, and what they were feeling. All of these opportunities to capture student thinking add to the chances of finding out what students are thinking and why. The reflections of the teachers (in class and in their post-lesson interviews) helped me to understand what the students had been exposed to previously.

Understandings of Nature of Rectangles

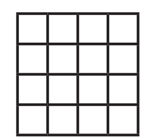
This task was undertaken in Grade 3/4, Grade 4, and Grade 4/5/6 classes. Teachers were confident that their students (in Grades 3-6) understood the nature of rectangles, and students (in Grades 4-6) understood what right angles were. The teachers (and myself) were surprised by some of the student responses and reflected upon them, as did other staff in whole school sessions where these findings were reported. Before I go any further, let me reemphasize that limited understandings of key ideas in mathematics occur across the world and are a major international research focus. The teaching and learning approach, and data collection methods used here allowed us to gain new insights into student understandings of the mathematical ideas that arose during this task (rectangles, right angles, factors, multiplication, and arrays).

During the first reporting session in Activity 2, the following comments were made as different groups each displayed some of the rectangles (see Figure 2) they made using 14 squares (1 × 14), 12 squares (3 × 4), and 16 squares (4 × 4) (see Figure 2). Similar comments were made in Grades 3-6; some by high performing students on their usual class mathematics tests. Comment c) was also made by a number of Year 8 students (in one of my previous studies).



a) "We found this one but it's not a rectangle—it's too long"

b) "That one is not a rectangle—it's too like a square"



c) "That one is not a rectangle, it's a square

d) When a group reported this square as a rectangle, another group responded: "No! A square is not a rectangle" in their later report.

Figure 2. Student reports to the class as they made the figures displayed

The teachers and I were astounded at first. The reflections of one of the teachers contributed significantly to the task I subsequently developed and describe later. I report the substance of the teacher comment rather than the exact wording because this comment was made quietly to me in class when the teacher realized the probable reason:

"Of course! This happens because of what we teach students in the Early Years. We show them a square [teacher made hand movements drawing a square] and say 'see ... this is a square because all of the sides are the same length'. Then over here we show them a rectangle [the teacher stepped sideways making hand movements in a space a considerable distance from where the square was 'drawn] with two long sides and two short sides [gesturing about half as long] and we say 'this is a rectangle ... see it has two long sides and two shorter sides' and so they think of that shape with the two long and the two half as long sides as the shape of a rectangle. We don't show them other shaped rectangles, and we don't connect the rectangle and the square."

Discussions with teachers in other primary schools in and external to my research study, and whole staff discussions in one research school confirmed this was so. What we have identified is that lack of variability in the shapes we use when exposing students to rectangles, seems to have contributed to students developing a narrow perception of what a rectangle is. Very long thin rectangles, rectangles with both pairs of sides close to the same length, and squares, are not seen by some students (including some high performing students) to be rectangles because they do not fit the visual image of a rectangle as it is commonly presented.

An understanding of rectangles in terms of their properties (two pairs of opposite sides equal in length and right angles at each vertex) was not a way many students thought about rectangles. Few students considered squares to be rectangles (possessing two pairs of opposite sides equal, and in addition all fours sides equal being a case of a special rectangle). Teacher reflections about what they had seen in various schools in which they had taught suggested school curricula do not tend to make explicit connections between squares and rectangles: they are commonly taught as different rather than related shapes.

The interview comment from a Grade 4 student (Fletcher, pseudonym) raises questions about what we deny our students, who prefer to make meaning of mathematics, when we present fragmented rather than connected ideas. Fletcher stated [in his interview]: "[Today] I have learnt what I always wanted to know!" He explained: "when we were doing shapes [in the past] I would think 'well what is the square related to?' and now I know it is relating to rectangles". He elaborated further when asked. The dots in the quote below indicate words omitted without changing the meaning and square brackets are added to

explain the context:

"Yeah because I always thought that a square didn't really fit into any real pattern of shapes ... a square ... never really had any ... (pause) family in shapes ... like a circle it has like an oval (pause) and a triangle ... it [can be] like a 'slopey' triangle [sketched as he talked], and triangles ... [can be] really thin"

Finding that a square did in fact have a family (see Figure 3) during discussions around the Shaping Rectangles Task changed Fletcher's way of thinking about mathematics:

"Now I am really intrigued by maths because before ... when maths was on I was just *alright* with it but now it really feels like *fun*"

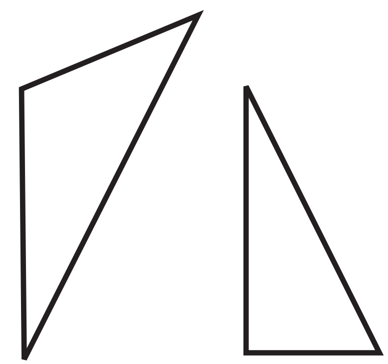


Figure 3a Members of triangle family

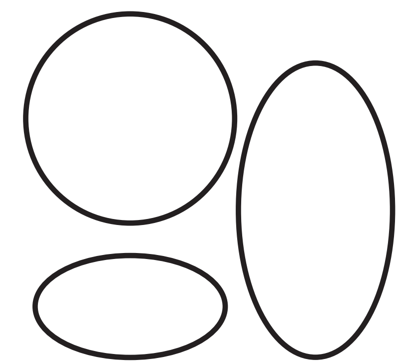


Figure 3b Members of circle family

After hearing Fletcher's comments showing his relief at finding squares do have families too, and his subsequent engagement, and observing students' reports showing limited understandings of rectangles (Figure 2) and right angles (Figure 4 discussed later), and trying the task I designed to overcome the limited understandings students displayed (Figure 6), teachers from this school began to think about where and how to modify approaches to overcome this problem of students seeing 'one visual image' of 'rectangle'. The prep teacher discussed taking down the poster showing only the conventional image for rectangle, and the square beside it as a different type of figure ('rectangle' and 'square'), and all staff discussed ways to add variation to the experiences students encountered in working with geometric figures as they progressed through the school.

Understandings of Nature of Right angles

When finding possible rectangles, and their features, interesting findings about student understandings of right angles arose. Across all classes studied, groups differed in the number of right angles they considered each rectangle possessed (one, two, three, or four). Figure 4 shows examples of group reports to the class, and one student comment after a reporting session showing disagreement with a position presented. Without group conversations, and reports to the class at intervals, these 'unexpected understandings' were unlikely to be exposed.

a) I thought there was only one right angle in a rectangle [at A]

b) No, that's a left angle [at D] not a right angle!

c) We weren't sure whether a right angle was the same where it was up the top [at B] or like that [at A]

d) We disagreed on whether a rectangle had four right angles or two [at A and C]

Figure 4. How many right angles different groups 'see' in rectangles

Why are students making such comments? We tend to assume they can easily see that a rectangle has four right angles—one at each vertex. What are the understandings that underpin student comments such as these? Quote a) in Figure 4, teachers tell me, is likely to be due to an absence in variation in the way right angles are drawn in text books, on posters, and on the board in class. Reflections from some teachers in the research study and research schools were of the following nature: "you know, we tend to draw right angles in an upright position with the lines pointing right, and up [like Figure 5]".

Figure 5. Conventional representation of a right angle

Quote b) in Figure 4 follows logically. If the image most commonly presented has a line pointing right, and is called a right angle, then if that line points left it is a left angle. Each

quote in Figure 4 shows an absence of understanding of the concept of angle as amount of turn from one line (ray) to the other? Student justification of the quote in d) is shown below to help think about this. Vertices as marked in Figure 4 are included in square brackets in the quote to indicate what was drawn/pointed at.

"We didn't know if there would be four right angles or just two because a right angle is … that [uses marker to draw a rectangle like Figure 4 on the board and highlight parts of lines AB and AD close to A and meeting at A], and umm, that part there [highlights right along line AD] … that would be just a continuing L shape, so there wouldn't be another right angle there [points to D] … so if that was the continuum [extends highlight along all of AB] that would be the one with the right angle [points to A], … I don't reckon, there is another one here [at D], but that means, there can't be another one there [at B by the same argument] … so there would be only two right angles …"

Again a lack of variation in images presented has contributed to some 'unusual understandings'. How can we build more meaningful understandings of the nature of rectangles, and right angles, and although not addressed here, the nature of angle generally? White (2001) provides useful advice in this area.

Increase Exposure to Variation and Highlight Relationships

Instead of the Shaping Rectangles Task starting with an activity to raise student awareness of the features of rectangles, a task was designed to raise student awareness of: variation in shapes that are rectangles, variation in orientation of angles that are right angles, and relationships between squares and rectangles. Terms were quickly defined in simply language with illustrations: quadrilaterals, right angle (using 'amount of turn'), and parallel lines. After some initial group experimentation with generating quadrilaterals to fit different categories, Figure 6 was presented and groups were asked: Can you make several very different figures to go in each part of this diagram? Can you add other sections in the diagram that contain other shapes that can be described using these same terms 'parallel', and 'right angles' (and possibly additional descriptions about sides)? And / Or can you write a set of instructions that would FORCE someone to make a rectangle?

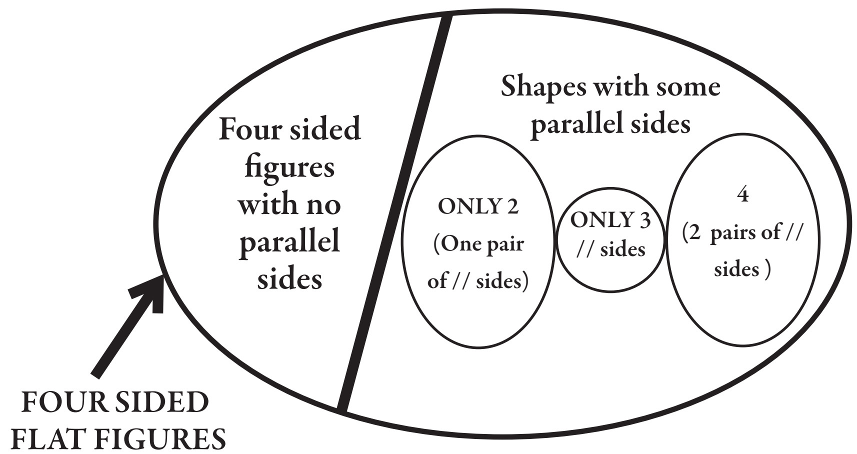


Figure 6. Diagram students were asked to add figures to

Several students who undertook this task commented: "I had never realised there were other sorts of four-sided shapes. I thought there were just squares and rectangles". This confirms the need to expose our students to variation.

Inserting this task instead of Activity 1 in Figure 2 assisted students to progress more smoothly through Activity 2 because the types of responses in a) and b) in Figure 2 did not occur, and those in c) and d) were far less frequent. The understandings students gradually developed as they worked with Activity 2 are listed here and illustrations of what occurred are found in my other article in this volume (Williams, in press).

Over three eighty-minute sessions students developed, to varying degrees, the following deep and connected understandings across topics. These include: a) How to find rectangles made with a given number of squares; b) Patterns linking factors, rectangle dimensions, and number of squares; c) Recognizing pairs of factors that multiply to the total number of squares are dimensions of the rectangles; d) Awareness of internal structure to arrangement of squares inside rectangles (arrays); e) Why pairs of factors that multiply to the total number of squares are dimensions of the rectangles (grouping through arrays); f) How to argue that all possible rectangles have been found; g) How to systematically generate all factors of a number.

These connect understandings also underpinning the areas of rectangles topic and some other area topics.

Conclusions

Students may develop more limited understandings than we expected if they are not exposed to sufficient variation in shapes, and allowed to explore and come gradually to their own understandings. Through the initial task described, limited understandings were exposed, and the task was refined to help overcome them. The deep and connected understandings that the refined task helped to develop show:

- What deep and connected understanding can look like
- Tasks and teaching approaches that can help to develop them

The connected understandings developed fit the goals of the Australian Curriculum—Mathematics (National Curriculum Board, 2009). The autonomous and creative student exploration identified as appropriate by the Melbourne Declaration on Educational Goals for Young Australians (Australia Ministerial Council on Education, Employment, Training, and Youth Affairs, 2008) fits with the Engaged to Learn approach used here. This research contributes to a 'map' of relationships students develop as they progress to deeper understandings for some of the problematic areas identified. We need to all work together to add detail to such a 'map' to help students develop such understandings in mathematics. By doing so, we will be more aware of what we need to look for and draw out in student talk. This paper also raises questions: What do high performing students really know when we accelerate through 'telling' rather elicit thinking through stimulating exploration. It also raises questions about what 'big ideas' students develop as they undertake these tasks. Can we go up another level and list fewer 'bigger ideas' to which the relationships identified contribute? And, in what other mathematical topics do we need to increase the variation to which students are exposed? What are the relationships they should be developing if they are going to understand deeply in that case? And what types of tasks could help to achieve this?

Acknowledgements

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References

Pegg, J. & Currie, P. (1998). *Widening the interpretation of van Hiele's Levels 2 and 3.* In A. Oliver & K. Newstead (Eds.), Proceedings of the 22nd Conference of the International Group for the Psychology of Mathematics Education, (Vol. 3, pp. 335-342). Stellenbosch, South Africa: PME.

National Curriculum Board. (2009). *Shape of the Australian Curriculum: Mathematics.* Retrieved August 22, 2010, from http://www.acara.edu.au/phase_ 1_-_the_ australian_curriculum.html

Ministerial Council on Education, Employment, Training, and Youth Affairs. (2008). *Melbourne declaration on educational goals for young Australians.* Retrieved August 22, 2010, from http://education.qld.gov.au/publication/ production/reports/future/ index.html

White, P. (2001). *An introduction to teaching calculus.* In L. Grimison and J. Pegg [Eds.]. Teaching Secondary School Mathematics: Theory into Practice. Victoria: Nelson Thompson Learning. 165-185

Williams, G. (in press). *Building resilience to build problem solving capacity: Tasks implemented for this purpose.* To be published in MAV Conference Book, Decemeber, 2010.

Williams, G. (2005). *Improving intellectual and affective quality in mathematics lessons: How autonomy and spontaneity enable creative and insightful thinking.* Unpublished doctoral dissertation, University of Melbourne, Melbourne, Australia. Accessed at: http://repository.unimelb.edu.au/10187/2380

Williams, G. (2003). *Student inclination to work with unfamiliar challenging problems: the role of resilience.* In B. Clarke, A. Bishop, R. Cameron, H. Forgasz & W. Seah (Eds.), Making Mathematicians (pp. 374-385). Melbourne, Victoria: Mathematical Association of Victoria.

BUILDING RESILIENCE TO BUILD PROBLEM SOLVING CAPACITY: TASKS IMPLEMENTED FOR THIS PURPOSE

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Where students have opportunity to work in groups with unfamiliar problems to find their own ways to solve them, they can achieve successes accompanied by feelings of excitement, satisfaction, and / or pleasure associated with what they have been able to achieve. Over time, this can build resilience, which should improve problem-solving capacity. This paper reports some such successes to illustrate how we might be able to build deep understandings and resilience simultaneously.

Introduction

The resilience-building features of a teaching and learning approach are illustrated through analysis of situations in which students gained mathematical 'successes'. Through these illustrations, features of the task, class interactions, and lesson structure become apparent as does the intensity of feelings that can accompany this process. These illustrations are drawn from my research into the role of optimism in collaborative problem solving where four cameras capture the interactions in groups, and group reports to the class as a whole (that occur at intervals during problem solving activity). In individual interviews after each lesson students use the video remote to find parts of the lesson that were important to them as they view a mixed video image of their group interactions and reports to the class.

Resilience Building

Martin Seligman (1995) showed that successes students achieved when working in 'flow' situations can build optimism (resilience). Optimistic students see their failures as temporary and able to be overcome through personal effort by looking into situations to find what they are able to change to increase their likelihood of success. They see their successes as permanent and as attributes of self (see Williams, 2009 for more information). Instead of perceiving what they were not able to do as 'failure', they perceive it as something yet to be achieved. Flow (Csikszentmihalyi, 1992) is a state of high positive affect that accompanies creative activity. Flow occurs during mathematical problem solving (Williams, 2002) when groups set themselves a mathematical challenge that is almost out of reach that requires them to build new skills and understandings to overcome it.

During flow, groups struggle to find ways to overcome challenges they set themselves and there is an intensity of focus. Overcoming the challenge involves developing mathematical ideas they did not previously possess. In other words, skills and understanding that are 'almost out of reach' are developed. The group selects the mathematics to use and decides how to use it. In other words, they engage in autonomous problem solving activity. Without groups spontaneously setting their own challenges, and autonomously working to overcome them, the conditions for flow would not exist.

The challenges students set themselves differ from group to group. There is opportunity for all students to engage in such activity if the task is accessible through multiple representations (e.g., numerical, tabular, diagrammatic, graphical, algebraic). As groups listen to and think about the reports of other groups they can begin to make links between various representations. Such linking of representations can lead to 'seeing' mathematical connections not realized previously and intense positive feelings can accompany these realizations.

Engaged to Learn Approach

The *Engaged to Learn Approach* I developed over time as a teacher, and refined through my research, requires the simultaneous engineering of several features to provide opportunities for students to engage in flow situations. These features include appropriate group composition, a complex task, a classroom environment providing emotional security whilst encouraging cognitive risk, and the fostering of conversations in small group and whole class settings that progressively elicit more complex thinking from students.

Tasks are undertaken over several sessions with the following sequence of activity:

1) Minimum mathematical information is given to introduce the task;
2) Small group brainstorming session is focused by teacher;
3) Groups work for 5-10 minutes on the task as teacher listens and asks questions;
4) Teacher makes general suggestions about what groups might report on;
5) Groups prime reporter: decide what to report, listen to and refine practice report;
6) While reporter is primed, teacher decides order of reporting: so all groups report something new;
7) Each group gives a 1.5 to 2 minute report;
8) Questions asked by the rest of the class must focus on clarifying or elaborating what was reported (not contradicting, or focusing outside what was reported);
9) After each report, teacher selects some aspect of the report that is of value to whole class, accents this contribution without affirming, and asks questions to stimulate further thinking;
10) Summary Time: teacher asks class whether there are further comments and students can query what was said (giving justification) if they did not get a chance to do so in their own report;
11) Teacher poses questions to stimulate further thinking in the next brainstorming session, and students are under no obligation to focus on these questions;
12) Cycle 2)-11) continues until teacher decides students have learnt sufficient (balancing this against time taken).

For flow situations to occur, the nature of the task is important, as are the types of comments teachers make and the types of questions they ask.

After the *Engaged to Learn Approach* has led to student generation of many ideas involved in the topic, the teacher summarizes the topic drawing on what students have developed. Students then undertake exercises in class or for homework (one of each type of problem for a start). Where students have difficulty, other students explain how to solve these problems on the board or where necessary, the teacher explains. Students then practice more of the exercises with a focus on areas in which they had difficulty.

Task

The initial sections of the Shaping Rectangles Task included students finding as many ('filled in') rectangles as possible using top surfaces of all of 14, then 12, then 16 square tiles (lying flat on the table) and discussing whether:

a) Using more tiles always produced more rectangles, and why; and

b) Developing an argument for why they knew they had all possible rectangles for a given number of tiles.

The final part of the task asked students to find the 'best number' between 1 and 46 (the number of tiles that would make the most rectangles). See Williams (in press) in this book for further elaboration of the task, and descriptions of some students' initial unusual 'understandings' about the nature of rectangles.

The Shaping Rectangles Task as implemented allowed students to experiment with concrete materials at the start of the task so all students had access to the task. As only 14 tiles were given, and these were all required to make each rectangle, students were less able to 'break away' from the group and experiment alone. Thus, task structure increased likelihood of groups working together. Only fourteen tiles were provided to encourage students to shift from trial and error with concrete aids to mental activity (when thinking about rectangles containing 16 tiles). By encouraging students to think beyond concrete aids, and giving groups opportunities to hear strategies of others, it was expected that students would have a broader variety of strategies to draw from in undertaking the final part of the task where they were required to consider rectangles containing far more tiles. This task is accessible at various levels, students could:

- Experiment with the concrete aids to try to make a rectangle
- Continue to experiment to find more rectangles
- Recognise patterns as they find possible rectangles
- Shift to numerical thinking about what numbers divide into the total if they found a pattern, and/or
- Use factors not concrete aids where they recognize their relevance.

Excitement About Factors

Stop for a minute and think about what your students would do. I have used this task with Grade 3 to Year 7 students in a variety of schools. Students generally start with trial and error rather than immediately recognizing the relevance of factors. Where students are not told what mathematics to use and how to use it, you gain a better understanding of what they 'really know' / can use. *And,* they become excited when they recognize what mathematics may be useful to them.

When asked to search for patterns, in general, groups do not initially 'see' that side lengths can always be divided a whole number of times into the total number of tiles in the rectangle. Seeing this causes some excitement, and checking to see if it is always so. Very few groups initially see a link to 'factors'. Table 1 is a composite of group responses used to illustrate how teachers can make decisions on reporting order, and on comments they make to value each group's contribution.

Table 1. Group reports in order given, with teacher's 'valuing' comment

No	Report Content	Teacher Comment
1	We found the number of tiles along sides of rectangles with 14 tiles divide into 14	What an interesting thing! I wonder does it always happen.
2	We found the same thing as Group 1 and we have started to check with the rectangles containing 12 tiles and we are finding the number of tiles along the side does divide into 12.	Notice what this group has done, not only have they linked their report to Group 1's report rather than say the same thing, they have also checked to see whether their pattern works with another number of tiles. This is what mathematicians do: check to see whether their pattern works in other cases.
3	We found what Group 1 and 2 found. Here is an example: [makes 3×4 rectangle]. See, 3 on the side and 3 goes into 12 four times (a whole number of times).	Ah! Now we have an example. It really helps when people give an example to explain what they have found rather than just say it.
4	We found the number of tiles on sides are factors of the total number of tiles.	This group have brought a new mathematical word into their report. They have not told us what it means. I wonder if another group will explain more.
5	We also found what Groups 1, 2 and 3 found. Like Group 4 we knew they were factors. Factors go into the number a whole number of times. When you multiply the number of tiles in the length and the number of tiles in the width together, you get the total number of tiles in the rectangle see 3×4; 2×6; 1×12 all make 12.	This group linked their report to what others have found, and explained the meaning of the new term they used. They have also found a new pattern. Your group can think about it and decide whether it always works. If so, I wonder why …?

These illustrations are drawn from the three classes in my research: Grade 3/4, Grade 4, and Grade 4/5/6 classes in schools in two School Sectors (Government, Catholic) in different regions of Melbourne (Northern, Southern). The initial understandings of students were similar across classes as were the ways in which their knowledge developed. This composite of what occurred across classes is intended to succinctly convey the types of things that happen. Table 1 describes group reports (generally from second reporting session) in the order reported, and types of comments made to value each contribution, and questions posed to stimulate further thinking. Each comment was followed by a thank you to the reporter/group but not clapping. Clapping includes an element of popularity as opposed to cognitive quality. I suggest clapping be delayed to the end of the task when they can congratulate the class as a whole. Internal feelings of success have generally 'taken over' from a need to clap by then.

Reporting order is important. For example, Group 1 would not have had anything new to contribute if they had reported later. For each set of reports, the teacher needs to decide whether to let the class ask questions after each report. When students are new to the process, you sometimes decide not to let students ask questions to keep focus, and limit time. Asking questions does add engagement though so it is a 'balancing act'. Questioning was not encouraged for the reports Table 1, because answers might 'bring out' what I wanted to value in the next report. For example, someone would probably have asked Group 4 about factors and Group 5 would not have gained credit for including such detail in their report. Each report gave opportunity for groups to share successes they achieved and this sometimes resulted in excited exclamations from others. This amplified these successes. Teacher ordering of reports, and valuing of contributions amplified group successes. Where the class exclaimed at something found, successes were amplified even further. This illustration highlights the teacher's role enabling simultaneous knowledge development and resilience building.

After the final report in Table 1, there is opportunity to increase student understanding of the mathematics involved whilst encouraging them to participate in the questioning process. For a start, they generally need quite a bit of encouragement but over time they become more comfortable asking questions. Possible teacher comments/questions to elicit student questions include: "Does anyone want to ask this group more to help everyone understand the meaning of 'factors'? [Long pause] Anyone …? [Long pause] I don't believe it! Does this mean I could ask anyone and they could tell me what factors are …? [Long pause]. You could even just ask them to repeat what they said …?"

This generally results in someone asking the reporter to repeat what they said, and then others generally begin to answer questions.

The teacher can then focus the next small group brainstorming session with comments such as: "Now you have found rectangles made using 12 tiles, start to think about whether you have them all and whether you can make a mathematical argument for why you have them all". The previous reports give groups access to arguments about having found factor pairs but not why factors are involved. To extend group thinking about this, the following type of comment could be made: "You might like to think about the patterns groups have found. When mathematicians find new patterns, they like to check they always work and then think about *Why?* Why do we get that pattern? Why does multiplying the number of tiles along this side and that side give us the total number of tiles used?" Note I have not introduced terms (length, width) unless students have already used them.

What Did They Know If They Knew Factors Were Involved?

We are now at one of the critical points in the task where we could assume students 'know it all': that rectangles are made up of arrays so you find a pair of factors of the total number of tiles and this tells you the length of a row in the array, and how many rows there are. In the next reporting session, students gave arguments about having all possible factors. Some generated factors in a systematic way showing there were no more. When discussing the 2×8 rectangle, they used '2×8' on the board and linked the 8 and the 2 to the number of tile lengths along the sides. They did not explain why the '×' was there even though I asked this question after each reporter used that notation. The following excerpts from student post-lesson interviews suggest students tend not to 'see' the arrays initially if they are not reminded they are there. In one post lesson interview, a student reported her surprise when she suddenly realized (towards the end of the lesson) *why* you multiply. The substance of her comment was: "We had made the 2×8 rectangle [she made one on the table with tiles] and Noel bumped the table and the two rows fell apart and I could see it was two groups of eight. I was so surprised! Suddenly I could see *why!"* In another school, after one of the later reporting sessions, a student described what she had heard about the 2×8 rectangle being made up of two groups of eight. As she demonstrated with the tiles, she suddenly exclaimed. The substance of her comment was: "Oh! Look! You can go the other way too … it's eight groups of two [separated rectangle into eight columns]". These are examples of mathematical insights accompanied by feelings of high positive affect. These insights included why multiplying length and width gave total tile number, and a 'start' to realization that multiplying in either order gives the same result.

Conclusions

When students are learning new mathematical topics, it can take time to 'see' what to us can seem obvious. There can be many ideas that together form a patchwork of connected mathematical relationships (as shown herein). My research has shown that struggling with unfamiliar problems can build deep understandings, and that intense positive feelings associated with such struggles can build resilience. Students who are not usually considered 'mathematically able' can make valuable contributions that give others a chance to explore further, and those non-resilient high performing students who are not inclined to explore may begin to do so.

Acknowledgements

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References

Csikszentmihalyi, M. (1992). The flow experience and its significance for human psychology. In M. Csikszentmihalyi & I. Csikszentmihalyi (Eds.), *Optimal experience: Psychological studies of flow in consciousness* (pp. 15-35). New York: Cambridge University Press.

Seligman, M. (with Reivich, K., Jaycox, L., Gillham, J.). (1995). *The Optimistic Child.* Adelaide: Griffin Press.

Williams, G. (2002). Associations between mathematically insightful collaborative behaviour and positive affect. A. Cockburn & E. Nardi (Eds.), *Proceedings of the 26th conference of the International Group for the Psychology of Mathematics Education* (Vol. 4, pp. 402-409). Norwich, UK: PME.

Williams, G. (2003). Student inclination to work with unfamiliar challenging problems: the role of resilience. In B. Clarke, A. Bishop, R. Cameron, H. Forgasz & W. Seah (Eds.), *Making Mathematicians* (pp. 374-385). Melbourne, Victoria: Mathematical Association of Victoria.

Williams, G. (2009). Engaged to learn pedagogy: Theoretically identified optimism-building situations. In R. Hunter, B. Bicknell, & T. Burgess. *Crossing Divides: Mathematics Education Research Group of Australasia 32 Conference Proceedings,* (Vol. 2. 595-602). Wellington, NZ: MERGA.

Williams, G. (in press). Australian Mathematics Curriculum Implementation: Teacher Reflections on Student Thinking Informing Future Directions. *Accepted for publication in the conference proceedings for the Mathematics Association of Victoria Annual Conference, December 2010.*

TRANSFORMING SPACE

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Space is an often neglected area of the curriculum. This workshop will focus particularly on activities which develop an understanding of transformations in Space and include reflections, rotations, translations and dilations as well as tessellations. A range of activites that are fun and often creative and artistic are included. Fun - with a focus on key mathematical ideas.

Introduction

The different aspects of the Space curriculum all connect together and interact so that one section cannot really be studied without connections to other areas. The major area of the Space curriculum that is the focus of this paper is transformations, but transformations cannot really be studied without reference to shape. Children intuitively use transformations very early in their work with shape, but often it is implicit rather than explicit. For example when children first experience triangles they are handling real triangles. They then start to be presented with pictures of triangles in books and these are often presented in a prototypical way with the third vertex above a horizontal line making a triangle that is isosceles or equilateral or even a right angled triangle. When presented with a triangle that has the third vertex below the line many children call it an "upside-down triangle" as they can see that rotated or reflected it would be a "triangle". I doubt any teacher has taught them that name.

There are three isometric (same measure) transformations, that is, transformations where the size and shape remain the same. These are translation, reflection and rotation (often called 'slide', 'flip' and 'turn' in early years though there is no reason the correct terms cannot be used). There are non-isometric transformations as well that children experience, particularly in an 'e-world', and these include dilation about a point and about a line as well as skews. Most children have experienced the stretching of pictures and subsequent distorting on the computer.

Activities

Copy cat 1

Materials: Pattern blocks, overhead projector (OHP) with OHP set of pattern blocks or pattern blocks on electronic whiteboard.

The main aim of this activity is to provide visualisation experiences and to encourage the students to attend to features of shapes and patterns as they work on visual memory. It would usually be used in grades 0-4 but could be used in more senior grades as a warm-up activity, a device to draw attention to aspects of shape or if the students had not previously had such experiences.

Make a shape using five pattern blocks (at least 3 different). Ask the students to make the same shape in front of them. Ask some of them to describe what they noticed about the shape (yellow hexagon, orange square, 2 red shapes (trapezium), 1 white rhombus; orange square was on the bottom etc.). Encourage the use of the correct names for the shapes but also challenge them with "How do you know?" or "What makes it a rhombus?" etc. Depending on how they go, repeat it with different shapes. If they manage to copy the shape while seeing it easily enough move to the next phase but if not, repeat with different shapes using some that are symmetrical and some that are not. Observe which students have difficulties.

Explain to the students that you are going to give them a challenge. You are going to show it to them for a short time during which they are not to touch the blocks. Then you are going to turn off the display and they are going to try to make the shape. They will need to try to make a picture of it in their minds. When they have made their attempts, tell them to put their hands on their shoulders and you are going to show it to them again. The students can look at it and look at the shape they have made and decide whether they need to change anything. Turn it off again and allow them time to make changes. Do not interrupt the students during this time as it is hard to keep information in your head when someone else talks to you.

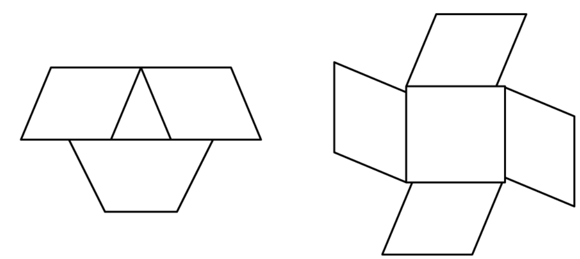
Start with four blocks only and a symmetric shape, then increase the number of blocks keeping it symmetric or start to use an asymmetric shape with fewer blocks. Increase the complexity as needed to provide a challenge to the class.

During the session use the correct names of the shapes (not "diamond" but "rhombus") and positional language. The students are making a translation of the shape they see but the shape names are a part of the activity as is the language of location (Shape and Location being the other two main areas of the space curriculum). This activity is really about visualising and visual memory as well and also assisting the students to pay attention to features of what they are observing.

Copy cat 2

As with the last activity this version uses pattern blocks. Students also need a sheet of paper and a ruler. A Mira (maths mirror which is both transparent and reflective) is also useful.

The main aim of this activity is to perform simple reflections in a mirror line and then to identify the axes of symmetry (mirror lines) in patterns. What is the meaning of symmetry? One misunderstanding that can occur is the confusion between reflectional and rotational symmetry. For example the two patterns below both have symmetry, but one only has reflectional symmetry while the other only has rotational symmetry. Too often rotational symmetry has not been discussed and children recognise the second pattern as having symmetry but it does not have a mirror line.



Misunderstandings occur early and, as with all concepts, seeing examples which are not part of the concept (or "non-examples") as well as those that are part of the concept is an important component of learning.

The first task is making a reflection in a mirror line. Start with a discussion of mirrors and have the students in pairs stand facing each other and take turns to move their hands and arms and be each others' reflection.

Students use one sheet of A3 paper between them with a vertical line drawn down the middle. One student arranges three pattern blocks on one side of the line. The other then makes the mirror image. The Mira can be used to check if the students are not sure. The roles reverse. Try with 4 blocks then 5. Then change the task so that the second person mirrors the first, then adds three blocks which goes back to the first person to mirror.

There is a difference between making a mirror reflection of a shape and detecting a line of symmetry within the one shape. Move to making a shape that has the central line as an axis of symmetry then making more symmetrical shapes with one axes of symmetry using 3, 4, 5, and more blocks

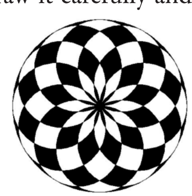
The challenge then is to add a horizontal mirror line and to make a pattern that has two mirror lines. Start using the language axes of symmetry. Challenge the students to make patterns that have 2, 3, 4 or 6 axes of symmetry (which the pattern blocks support well)

Copy cat 3

Moving to rotation - For this activity the students need to be sitting in pairs facing the same way. To start with one student makes a pattern with three blocks. The second student makes the same pattern then rotates the paper through a half turn and they both look at the pattern. How is it different and how the same. Ask the students to repeat the task with three and four blocks only, using both patterns with an axis of symmetry and patterns which do not have an axis of symmetry. Tell them that you want them to notice the differences and how to make it because you are going to challenge them to make one WITHOUT TURNING it. Lead a discussion on what they noticed. Then set one up on the OHP or whiteboard and challenge them to make the rotated version without making it first and turning it. Try a few then have the students challenge each other.

The activity now moves to making a shape that has rotational symmetry as opposed to visually rotating a shape already made. Make a shape with 5 blocks on the OHP that has rotational symmetry of 2 and challenge them to make it rotated through a half turn as they have been doing. They will realise that it is the same in both positions. Explain that this shape has rotational symmetry of 2 in that there are two positions in a full rotation where it is exactly the same. Ask them to make a shape with the blocks that has some rotational symmetry. Some of the shapes they make will have reflectional symmetry as well. Show some of the shapes made and work with the students to identify all the symmetries.

Introduce a shape pattern built around an equilateral triangle that has a rotational symmetry of three. Challenge the students to make a pattern then draw it carefully and identify all its symmetries. The final challenge is to make shapes that would fit into the cells of the table below and to see how many they can make (and draw to record their patterns). Not all are possible but if a student decides one is not possible, they need to be able to argue why it is not. The pattern blocks allow easily for symmetries



of 2, 3, 4, and 6. A further challenge is to use a compass and ruler to create a shape with many symmetries and identify them. The one drawn here above was created with a compass and ruler.

		Has no rotational symmetry	Has rotational symmetry of			
			1	2	3	4
Number of axes of symmetry	0					
	1					
	2					
	3					
	4					
	5					

The alphabet

This can be used as an introduction to reflection and rotation or as a revision type activity. It uses a set of Arial capital letters about 600 point font.

Do not tell the class the mathematics topic as that gives the game away. Tell the class that you are going to sort the letters according to some predetermined rule and you want them to try to decide what the rule is. Slowly sort according to symmetries. S, N and Z have rotational symmetry but no reflection (but in fact the S is just slightly larger at the bottom so … that will be a later discussion). O, I and X have both and I have always done a bit of an act – "Will I put it here? (with the other reflective symmetry letters) … or here? (with the rotationally symmetric letters)? I think I need to make a new group in between." When there are some in each of the 4 groups ask the students to see if they can predict where the next letter will go. Then ask them what they notice about the letters in each group – is there anything the same about them? They will notice things that are the same about pairs but may have more difficulty finding what it is about the whole group. I have always put the letters with one axis of symmetry together even if some have a vertical axis and some a horizontal axis. Later the students may separate the groups according to their own criteria.

When they have "discovered" the rule send them in pairs to the computer to explore the different fonts. For each font they explore, have the students sort the letters under the headings in the following table

Font name	Have no symmetry	Have rotational symmetry only	Have one vertical axis of symmetry	Have one horizontal axis of symmetry	Have both reflectional and rotational symmetry

As a follow up to this task, hub caps from cars provide a rich source of symmetries, particularly rotational though the “nuts” may need to be ignored.

Federation Square triangles

The triangles that make the walls of Federation Square are based on a simple two by one rectangle. For this task the students can explore how triangles fit together to tessellate then how the Federation square triangles fit. They will need a sheet of triangles, preferably about 6 cm by 3 cm as a minimum, to cut out and a large sheet on which to record their findings. Glue may be useful for this task.

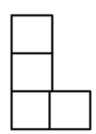
Start with the idea of tiling a floor and fitting the tiles together. Have the students cut out the triangles then make a floor tiling pattern with them, recording their pattern.

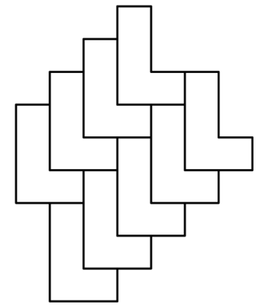
Now investigate how the triangles fit. Challenge them to try to make a larger triangle with 2 triangles. Can they make a larger triangle with three triangles? How about four triangles? Finally, can students make a larger triangle with five triangles? This larger triangle with 5 triangles is the basis of the Federation Square tiles. While all triangles can have four placed together to make a larger triangle the same shape (and the pattern is the same), only this triangle has five that can make a larger triangle that is exactly the same shape (though a different size hence it is called similar).

L tiles

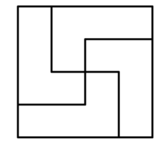
This activity also explores floor tiling patterns – in this case the tile is an L shape. Each student will need a sheet of Ls. Cut the Ls from the sheet. The challenge is to work as a group to see how many different tiling patterns you can make with these tiles. This is not just about making the tiling patterns but also about identifying the reflections, rotations and translations within the patterns.

There are some interesting and surprising tessellations. The L shape, made of four joined squares, tessellates in 17 different patterns. One challenge may be to see how many of these different tessellation patterns students can find. Some are shown here with the transformations identified.

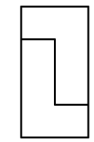




This one involves just a translation



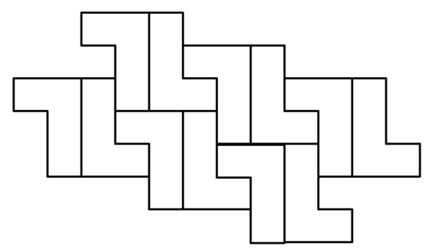
Four 90° rotations give a square which then tessellates



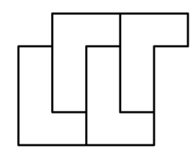
A 180° rotation creates this tessellating rectangle

Tesselations

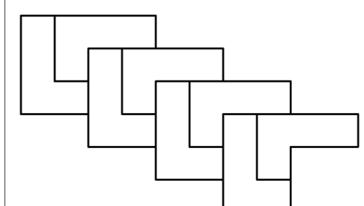
In this activity the process of making unusual shaped tiles that tessellate will be explored and participants make their own tessellating patterns. These patterns are based on translation, reflection and rotation and all start with a simple tessellating shape which is then transformed using translation, reflection and rotation.



A 180° rotation and translation of the new unit tessellates



A tessellation formed by a reflection in a horizontal mirror and a translation



A rotation through 90° followed by a translation also tessellates

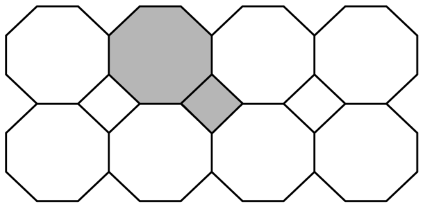
This is background on tessellations closely related to the activities.

A shape can be said to tessellate if it could be used as a floor tile and completely tile the floor with no other tiles needed. Cutting tiles for the boundary is allowed but the tiles should abut in all other positions without cutting.

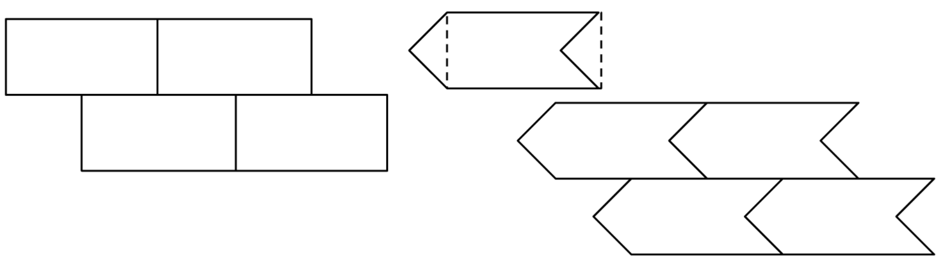
It is quite clear that squares, rectangles and triangles will tessellate.

Regular hexagons also tessellate creating the beehive pattern.

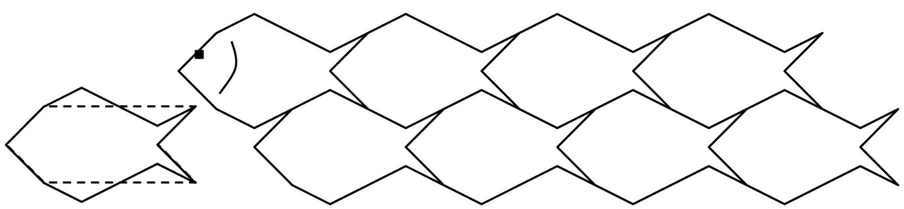
Regular octagons will not. A square "hole" is left. However the shape coloured in the diagram below will tessellate.



The really interesting tessellations are formed by altering a basic tessellating shape. Take a simple rectangle for example. We know it will tessellate in a brick pattern. If we alter one end of the rectangle, then translate that change to the other end, it will still tessellate as a row will still fit by a translation.



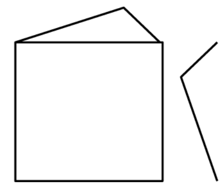
Using the same idea we can alter the top and bottom, but this time the shape has to be translated to the right as well as from the top to the bottom.



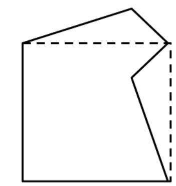
The fish was made just by using straight lines but there is no reason why curves cannot

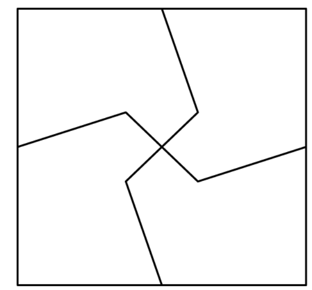
be used. The fish was also created using the draw menu of Microsoft Word. It is quite powerful and enable tesselating units to be easily translated.

The fish was also made by just using translation. Rotation is another possibility. Consider a rotation through 90 degrees starting with a square. Alter one side of the square, then rotate that alteration through 90 degrees to another side, making sure that the area stays the same.



Alter one side then copy it and rotate it 90° anticlockwise, placing it on the right of the square.





These new shapes will then rotate to fit together making a new larger square

There are many interesting computer programs that build tessellations. One that offers some good material is Tesselmania. The most famous artist who used the idea of tessellations was the graphic artist Mauritz C Escher. There are programs on CD and on the web that present his works along with explorations of the creation of tessellations.

Students can be quite creative in their tessellating tiles as in the case of the pirates below, created by some grade six children (but I cannot remember the source).